

Fredholm Theory and Stable Approximation of Band Operators and Their Generalisations

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von Dr. Marko Lindner,
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Chapter 1

Introduction

This text is written to summarise my research activities over the last years. Parts of this research have been summarised before in the monographs “Infinite Matrices and Their Finite Sections: An Introduction to the Limit Operator Method” [106] in 2006 and “Limit Operators, Collective Compactness, and the Spectral Theory of Infinite Matrices” [39] in 2008, the first single-authored and the latter co-authored by Simon Chandler-Wilde from the University of Reading, UK. The current text is therefore naturally a hybrid between these two monographs enriched with more recent results, both published and so far unpublished ones.

Classes of infinite matrices. The main theme of the body of work to be presented here is the Fredholm theory of bounded linear operators generated by a class of infinite matrices (a_{ij}) that are either banded or have certain decay properties as one goes away from the main diagonal. In the simplest case to be considered, the indices i and j run through the integers \mathbb{Z} and the matrix entries a_{ij} are complex numbers. Under certain conditions on the entries a_{ij} , the matrix (a_{ij}) then induces, via matrix-vector multiplication, a linear operator A on the space $E = \ell^2(\mathbb{Z}, \mathbb{C})$ of two-sided infinite complex sequences with absolutely summable squares. We call A a *band operator* if (a_{ij}) is a band matrix with uniformly bounded entries, and we call it a *band-dominated operator* if it is the limit, in the operator norm on E , of a sequence of band operators. The set of all operators A whose matrix (a_{ij}) has a summable off-diagonal decay, that means

$$\sum_{k \in \mathbb{Z}} \delta_k < \infty \quad \text{with} \quad \delta_k = \sup_{j \in \mathbb{Z}} |a_{j+k,k}|,$$

is called *the Wiener algebra*. This is a particularly nice class of bounded linear operators containing all band operators. A matrix with this property generates a bounded, and in fact band-dominated, linear operator on all spaces $\ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$.

Fredholmness and limit operators. For the Fredholm theory of a band-dominated operator A , the values of any finite collection of matrix entries a_{ij} (say $\{a_{ij} : -100 \leq i, j \leq 100\}$) is completely irrelevant as changing these values only perturbs A by a finite rank operator. It is therefore clear that the key to the Fredholm properties of A is to understand the behaviour of the entries a_{ij} as $(i, j) \rightarrow \infty$. Since we generally do not assume convergence of our matrix entries at infinity, this asymptotic behaviour¹ cannot be reflected by a single number; it has much more complexity and needs a more involved storage device: the so-called limit operators of A , each of which is an operator on E itself. Precisely, with every sequence $h = (h_1, h_2, \dots) \subset \mathbb{Z}$ going to infinity for which the sequence of matrices

$$(a_{i+h_k, j+h_k})_{i,j}, \quad k = 1, 2, \dots \quad (1.1)$$

converges entrywise as $k \rightarrow \infty$, we associate the operator that is induced by the limit of this matrix sequence (1.1) and call it the *limit operator* of A with respect to h , denoted by A_h . The collection of all limit operators of A is denoted by $\sigma^{\text{op}}(A)$; it carries all the information about the Fredholm properties of A . In fact, one can show [96, 140] that a band-dominated operator A is a Fredholm operator, in which case its Fredholm index can be calculated [139] by looking at two members of $\sigma^{\text{op}}(A)$, if and only if all members of $\sigma^{\text{op}}(A)$ are invertible and their inverses are uniformly bounded. If A is even in the Wiener algebra then this uniform boundedness condition can be dropped, yielding the formula

$$\text{spec}_{\text{ess}} A = \bigcup_{A_h \in \sigma^{\text{op}}(A)} \text{spec } A_h \quad (1.2)$$

for the essential spectrum of A in terms of the spectra of its limit operators.

More general spaces. Many of these ideas generalise to the case when A acts on $E = \ell^p(\mathbb{Z}^N, X)$, where $p \in [1, \infty]$, N is a natural number and X is a complex Banach space. The elements of E are functions $\mathbb{Z}^N \rightarrow X$, thought of as generalised sequences $(u_k)_{k \in \mathbb{Z}^N}$ with values in X , such that $\|u_k\|_X^p$ is summable over \mathbb{Z}^N . In this setting we are interested in band-dominated operators A that are induced by a matrix $(a_{ij})_{i,j \in \mathbb{Z}^N}$ with operator entries $a_{ij} : X \rightarrow X$. If $\dim X = \infty$ then the Fredholm theory of A changes; now a single matrix entry a_{ij} , being an infinite-dimensional operator itself, can change the Fredholm properties of A .

¹What we call “asymptotic behaviour” here is actually the coset of the matrix (a_{ij}) modulo the ideal K , where K is the closure, in the operator norm, of the set of all such matrices with only finitely many non-zero entries. In the setting of $E = \ell^2(\mathbb{Z}, \mathbb{C})$, this ideal K exactly corresponds with the compact operators on E , and the Fredholm property of A is equivalent to the invertibility of the coset $(a_{ij}) + K$ in a suitable factor algebra.

One can however prove an analogous theorem as before:

Under an additional condition on the band-dominated operator A , the coset $(a_{ij}) + K$ is invertible, in which case we now call A *invertible at infinity*, if and only if all limit operators of A are invertible with their inverses uniformly bounded.

(1.3)

The definition of a limit operator and of the ideal K hereby generalise literally from (1.1) and footnote 1 in the simpler setting above. The additional condition on A is that every sequence $h = (h_1, h_2, \dots) \subset \mathbb{Z}^N$ with $|h_k| \rightarrow \infty$ is required to have a subsequence g such that the limit operator A_g exists, in which case we call A a *rich* operator. This condition can be understood as a compactness property of the set of all translates of A ; it ensures that A has sufficiently many limit operators to establish the ‘if’ part in statement (1.3). In the case when $\dim X < \infty$ this condition is unnecessary because, due to the fact that bounded subsets of finite-dimensional spaces are relatively compact and by a standard diagonal procedure, every band-dominated operator is automatically rich then.

Depending on the space E (i.e. the choice of p and X) and the operator A , invertibility at infinity relates more or less closely to Fredholmness of A . Interestingly, it also relates to a completely different problem: the stability of certain approximation methods, for example the approximation of an infinite matrix $(a_{ij})_{i,j \in \mathbb{Z}^N}$ by the sequence of finite matrices $(a_{ij})_{i,j \in \{-n, \dots, n\}^N}$ with $n = 1, 2, \dots$. As a consequence, we can study particular problems in operator theory as well as in numerical analysis in terms of the central question whether or not all limit operators of a certain operator A are invertible with a uniform bound on the inverses, i.e. whether

- (C1) all elements of $\sigma^{\text{op}}(A)$ are injective,
- (C2) all elements of $\sigma^{\text{op}}(A)$ are surjective, and
- (C3) there is an upper bound M such that $\|B^{-1}\| < M$ for all $B \in \sigma^{\text{op}}(A)$.

In [104, 106] there is a particular focus on the case $p = \infty$, where it was shown, using a compactness property of $\sigma^{\text{op}}(A)$, that condition (C3) is always a consequence of $\{(C1), (C2)\}$ if A is rich. An even further simplification of this set of conditions for the same case $p = \infty$ was achieved more recently in co-operation with Chandler-Wilde by means of collectively compact operator theory.

Collective compactness. A family \mathcal{K} of bounded linear operators on a Banach space Z is called *collectively compact* if, for any sequences $(K_n) \subseteq \mathcal{K}$ and $(z_n) \subseteq Z$ with $\|z_n\| \leq 1$, there is a subsequence of $(K_n z_n)$ that converges in the norm of Z . It is immediate that every collectively compact family \mathcal{K} is bounded and that all of its members are compact operators. In [46] Chandler-Wilde and Zhang generalise the theory of collectively compact operator families

(that originally goes back to Anselone and co-workers, e.g. [4]) \mathcal{K} by requiring the convergence of a subsequence of $(K_n z_n)$ in Buck's [25] strict topology only. Now \mathcal{K} may contain operators with merely local compactness properties such as integral operators over \mathbb{R}^N with a continuous or weakly singular kernel. Chandler-Wilde and Zhang show the following collective version of Fredholm's alternative: If \mathcal{K} is generalised collectively compact in the above sense and some additional conditions (including the existence of a dense² subset \mathcal{K}' such that $I + K$ is surjective for all $K \in \mathcal{K}'$) hold on \mathcal{K} then injectivity of all $I + K$ with $K \in \mathcal{K}$ implies their invertibility and uniform boundedness of the inverses.

Collective compactness meets limit operators. In [39] we have shown that this result of [46] can be applied to $\mathcal{K} = \sigma^{\text{op}}(K)$ if $p = \infty$ and if K is band-dominated, rich and its matrix entries form a collectively compact set of operators on X . In this case it turns out that the set of conditions $\{(C1), (C2), (C3)\}$ on $A = I + K$ reduces to $\{(C1), (C2')\}$, where $(C2')$ is $(C2)$ restricted to a dense² subset of $\sigma^{\text{op}}(A)$. In the case when the matrix of K has almost periodic or pseudoergodic (in the sense of Davies [51]) diagonals or even generally for bounded diagonals in the case $N = 1$, one can show [38, 39] that also condition $(C2')$ is redundant so that $\{(C1), (C2), (C3)\} = \{(C1)\}$. This further simplification is extremely helpful in applications; moreover, in problems from mathematical physics, the injectivity in $(C1)$ can sometimes be established directly via energy or other arguments (e.g. [30]). The remaining condition $(C1)$ is often [169, 170, 94, 95, 38, 39] referred to as Favard's condition after Jean Aimé Favard's pioneering work [62] in the story of limit operators. A detailed account on both the history of limit operators and collectively compact operator theory can be found in the introduction of [39].

Due to the close connection, as established in [106, 39], between invertibility at infinity and Fredholmness for operators $A = I + K$ of the discussed form acting on $\ell^\infty(\mathbb{Z}^N, X)$, the above results yield new and simplified Fredholm criteria and therefore a new formula for the essential spectrum of such operators. Already in the simplest case, when A is an arbitrary band-dominated operator on $\ell^\infty(\mathbb{Z}^1, X)$ with $\dim X < \infty$, we get that A is a Fredholm operator if and only if condition $(C1)$ holds, and consequently we have the following modification of formula (1.2):

$$\text{spec}_{\text{ess}} A = \bigcup_{A_h \in \sigma^{\text{op}}(A)} \text{spec}_{\text{point}}^\infty A_h, \quad (1.4)$$

where $\text{spec}_{\text{point}}^\infty B$ denotes the point spectrum (set of eigenvalues) of an operator B on $\ell^\infty(\mathbb{Z}^N, X)$. If A is even in the Wiener algebra (for example a band operator) then A is bounded on all spaces $\ell^p(\mathbb{Z}, X)$ with $p \in [1, \infty]$, and one can show [107] that its Fredholm property (including the index) and hence its essential spectrum does not depend on p , so that (1.4) holds on all these spaces.

²with respect to strong, meaning pointwise, convergence in the strict topology

The benefit of formula (1.4), compared to (1.2), is that eigenvalues with respect to ℓ^∞ can often be found analytically. This is demonstrated in recent work on the spectrum of random (and therefore almost surely pseudoergodic) matrices that appear in so-called non-self-adjoint Schrödinger equations describing the propagation of a particle hopping randomly on a 1-dimensional lattice. The corresponding infinite matrices are supported on two diagonals only. Depending on the concrete application, these are either the sub- and main diagonal [24, 63, 64, 65, 109, 175] or the sub- and superdiagonal [64, 84], the entries of which are typically random samples from $\Sigma = \{-1, 1\}$ or from another compact set $\Sigma \subset \mathbb{C}$. In this case, the set $\sigma^{\text{op}}(A)$ almost surely consist of all infinite matrices with the corresponding random diagonal replaced by any sequence over Σ . In particular, $A \in \sigma^{\text{op}}(A)$ whence formula (1.4) not only gives the essential spectrum but also the spectrum of A considered as operator on any space $\ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$.

Approximation methods. We have already mentioned briefly that certain problems of numerical analysis can be reduced to the invertibility at infinity of an associated operator. If a band-dominated operator A on $E = \ell^p(\mathbb{Z}^N, X)$, induced by an infinite matrix (a_{ij}) , is invertible then, for every right-hand side $b \in E$, the equation $Au = b$ has a unique solution $u \in E$. To find this solution, one often replaces the infinite system $Au = b$ by the sequence of finite quadratic systems

$$A_n u_n = b_n, \quad n = 1, 2, \dots \quad (1.5)$$

where $A_n = (a_{ij})_{i,j \in \{-n, \dots, n\}^N}$ is the so-called n th finite section of the infinite matrix $(a_{ij})_{i,j \in \mathbb{Z}^N}$ and b_n is the respective finite subvector of the right-hand side $b \in E$. The hope behind this procedure is that, given $Au = b$ is uniquely solvable, also (1.5) is uniquely solvable (at least once n is big enough) and the solution u_n approximates the exact solution u componentwise as $n \rightarrow \infty$. This procedure is called the *finite section method* (FSM). One can show that this method works as desired – we will call it *applicable* then – if and only if A is invertible and (A_n) is *stable*, the latter meaning that all finite matrices A_n with a sufficiently large index n are invertible and their inverses are uniformly bounded.

So the key question is about the stability of the sequence (A_n) . The trick is as follows: One treats each A_n , after extending it by the identity operator, as an operator on E and stacks infinitely many copies of E , together with the operators A_1, A_2, \dots acting on them, into the $(N+1)$ th dimension. What results is a direct sum $\oplus A_n$, acting on $E' = \ell^p(\mathbb{Z}^{N+1}, X)$, of our operators. It is readily seen that the sequence (A_n) is stable if and only if the operator $\oplus A_n$ is invertible at infinity. The latter can be equivalently characterised in terms of limit operators of $\oplus A_n$ on E' which ultimately boils down to looking at limit operators of A on E . There are however operators A (and it is very simple to give such examples) where the FSM clearly fails to be stable. In this case we propose two different strategies:

A closer look at the construction of $\oplus A_n$ in the case $N = 1$ shows that, in the case of the FSM, the stability of each subsequence of (A_n) is governed by the behaviour of a corresponding subset of limit operators of A . This fact was first spelt out in [145] and was generalised to arbitrary dimensions N in [110], where also the choice of the cut-off geometry is discussed. The message is that in some cases where the whole finite section sequence (A_n) of A is not stable, one can, by changing the geometry and/or closely looking at the set of all limit operators of A , single out subsequences (A_{n_k}) that are stable and hence can be used for the approximate solution of $Au = b$.

Completely independent of this line of thought, there is the idea of working with rectangular rather than quadratic finite submatrices of (a_{ij}) if the latter raises problems. The idea is common practice in the numerical community and it goes back at least to Cleve Moler and the 1960s, where it was suggested in the ℓ^2 setting, instead of solving the quadratic system (1.5) exactly, to take over-determined rectangular subsystems of $Au = b$ and to solve them approximately by least squares. Together with Heinemeyer and Potthast [82], we have derived accurate theorems that prove this observation for operators A on $E = \ell^p(\mathbb{Z}^N, X)$ with $p \in [1, \infty)$ and for more general classes of Banach spaces E . Precisely, it was shown that, for every $\varepsilon > 0$, there exist $m_0, n_0 \in \mathbb{N}$ and a precision $\delta > 0$ such that all δ -approximate solutions of the rectangular system $A_{m,n}u_{m,n} = b_m$ with $m > m_0$ and $n > n_0$ are in the ε -neighbourhood of the exact solution u of $Au = b$. The downside of this method is that one still has to understand how to couple the matrix dimensions m and n with each other (for example, if A is a band operator with band-width w then it suffices to choose $m := n + w$). The attraction of the method, however, is the following: For the FSM of an invertible band-dominated operator A , one has applicability if and only if a couple of conditions hold for the limit operators of A . (Note that those conditions, and even the condition whether or not A is band-dominated, can sometimes be hard to check.) For the rectangular method, the stability analysis of [82] shows that the method always works as soon as A is invertible – and this even holds for operators A under the much weaker condition that the entries of (a_{ij}) tend to zero in each column, that is $\|a_{ij}\| \rightarrow 0$ as $|i| \rightarrow \infty$, for each j .

The motivation for the paper [82] was the numerical solution of boundary integral equations for 3D rough surface scattering problems. This is a delicate problem as these integral operators on $L^2(\mathbb{R}^2)$ are so-called ‘rough’ operators, with oscillatory kernels, rather than standard Calderon-Zygmund operators. It was not clear whether these operators are band-dominated on $L^2(\mathbb{R}^2) \cong \ell^2(\mathbb{Z}^2, L^2([0, 1]^2))$ (only that they are not in the Wiener algebra) and even their boundedness as operators on $L^2(\mathbb{R}^2)$ is far from obvious. So it is unclear whether the FSM is applicable which is why another method in a more general framework was needed.

In a similar area, together with Chandler-Wilde [36, 37], we have studied Fredholmness and applicability of the FSM for a class of boundary integral equations that models a variety of concrete physical problems such as free surface water wave problems and 2D rough surface scattering problems. In these applications, the integral kernel decays sufficiently fast for the operator to belong to the Wiener algebra, which enables us to use the limit operator machinery described above. A key ingredient of this work is to identify the class of integral operators with a Banach algebra generated by products of convolution and multiplication operators.

Organisation of the text. In Chapter 2 we introduce the fundamental objects of our studies; besides basic conventions and notations, there are, most importantly, the sequence spaces E and the strict topology on them. In Chapter 3 we look at the classes of operators $E \rightarrow E$ under consideration. These are classified in terms of continuity and compactness properties but also from the perspective of the corresponding infinite matrices. In Chapter 4 we come to the key tools: collectively compact operator theory and the theory of limit operators. By bringing these two together, we derive many of our results on Fredholmness and invertibility at infinity, most of which come to bloom in the setting of band-dominated operators in Chapter 5. Apart from Fredholm theory, the other branch of research that is followed here is stable approximation of our infinite matrix operators by finite matrices. This is the subject of Chapter 6, where the finite section method for band-dominated operators is studied and where we give two alternative strategies if the former fails, namely passing to subsequences and working with rectangular rather than quadratic submatrices. The largest and final chapter is Chapter 7, where we apply the theoretical results of Chapters 5 and 6 to concrete problems from mathematical physics such as discrete Schrödinger operators, random Jacobi operators for the study of e.g. quantum particles, and boundary integral equations modelling e.g. free surface water wave problems and 2D and 3D wave scattering problems by an unbounded rough surface.

Concerning the style of the text, I have tried to write a sufficiently detailed but still accessible exposition. For the sake of readability, I have restricted myself to the currently relevant rather than the most general case at several points. For example, in Chapters 2–4 we restrict ourselves to the sequence spaces E , which are the ones we need in Chapters 5–7, although the theory in Chapters 2–4 is available for more general Banach spaces E (see [39]). Another example is the restriction to approximation methods (A_n) with index $n \in \mathbb{N}$ in Chapter 6 rather than studying those with a continuous index set like $(0, +\infty)$, although stability of the latter has been established in [106].

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Chemnitz, February 2009

Marko Lindner

Chapter 2

Preliminaries

2.1 Numbers and Vectors

As usual, by \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} we denote the sets of natural, integer, rational, real and complex numbers, respectively. The positive half axis $(0, +\infty)$ will be abbreviated by \mathbb{R}_+ , the set of nonnegative integers $\{0, 1, \dots\}$ is \mathbb{N}_0 , and the unit circle in the complex plane; that is $\{z \in \mathbb{C} : |z| = 1\}$, is denoted by \mathbb{T} .

For every vector $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, we put $|x| := \max(|x_1|, \dots, |x_N|)$, and for two sets $U, V \subset \mathbb{R}^N$, we define their distance by

$$\text{dist}(U, V) := \inf_{u \in U, v \in V} |u - v|.$$

The decision for the maximum norm in \mathbb{R}^N implies that $|x|$ and $\text{dist}(U, V)$ are integer if $x \in \mathbb{Z}^N$ and $U, V \subset \mathbb{Z}^N$, which we will find convenient for the study of band operators, for example. Moreover, balls $\{x \in \mathbb{R}^N : |x| \leq r\}$ in this norm are just cubes $[-r, r]^N$, which will sometimes simplify our notation. However, since in \mathbb{R}^N all norms are equivalent, all of the following theory, apart from a slight modification of what the band-width of a band operator is, also holds if we replace the maximum norm by any other norm in \mathbb{R}^N .

For a real number $x \in \mathbb{R}$, we let

$$[x] := \max\{z \in \mathbb{Z} : z \leq x\}$$

denote its integer part. Without introducing a new notation, put

$$[x] := ([x_1], \dots, [x_N])$$

for a vector $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, so that $x - [x]$ is contained in the hypercube $H := [0, 1)^N$ for all $x \in \mathbb{R}^N$.

2.2 Banach Spaces and Banach Algebras

In this text the letter X usually stands for a complex Banach space; that is a normed vector space over the complex numbers which is complete in its norm. For brevity, we will often simply call this a *Banach space*.

When talking about a *Banach algebra*, we always mean a unital complex Banach algebra; that is a Banach space \mathcal{B} with another binary operation \cdot which is associative, bilinear and compatible with the norm in \mathcal{B} in the sense that

$$\|x \cdot y\| \leq \|x\| \|y\| \quad \text{for all } x, y \in \mathcal{B},$$

where in addition, we suppose that there is a *unit element* e in \mathcal{B} such that $e \cdot x = x = x \cdot e$ for all $x \in \mathcal{B}$. Note that in this case, the norm in \mathcal{B} can always be chosen such that $\|e\| = 1$, which is what we will suppose from this point.

As usual, we abbreviate $x \cdot y$ by xy , and we say that $x \in \mathcal{B}$ is *invertible* in \mathcal{B} if there exists an element $y =: x^{-1} \in \mathcal{B}$ such that $xy = e = yx$.

Moreover, when talking about an *ideal* in a Banach algebra \mathcal{B} we always have in mind a two-sided ideal; that is a subspace \mathcal{J} of \mathcal{B} such that $bj \in \mathcal{J}$ and $jb \in \mathcal{J}$ whenever $b \in \mathcal{B}$ and $j \in \mathcal{J}$.

If \mathcal{B} is a Banach algebra and M is a subset of \mathcal{B} , then $\text{alg}_{\mathcal{B}}M$, $\text{closalg}_{\mathcal{B}}M := \text{clos}_{\mathcal{B}}(\text{alg}_{\mathcal{B}}M)$ and $\text{closid}_{\mathcal{B}}M$ denote the smallest subalgebra, the smallest Banach subalgebra and the smallest closed ideal of \mathcal{B} containing M , respectively.

As usual, for a closed ideal \mathcal{J} in a Banach algebra \mathcal{B} , the set

$$\mathcal{B}/\mathcal{J} := \{b + \mathcal{J} : b \in \mathcal{B}\}$$

with operations

$$(a + \mathcal{J}) + (b + \mathcal{J}) := (a + b) + \mathcal{J}, \quad \|b + \mathcal{J}\| := \inf_{j \in \mathcal{J}} \|b + j\|, \quad a, b \in \mathcal{B}$$

is a Banach algebra again, referred to as the *factor algebra of \mathcal{B} modulo \mathcal{J}* .

A Banach subalgebra \mathcal{B} of a Banach algebra \mathbf{A} is called *inverse closed* in \mathbf{A} if, whenever $x \in \mathcal{B}$ is invertible in \mathbf{A} , also its inverse x^{-1} is in \mathcal{B} .

If \mathcal{B} is a Banach algebra and $x \in \mathcal{B}$, then the set

$$\text{spec}_{\mathcal{B}}x := \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{B}\}$$

is the *spectrum of x in \mathcal{B}* . Spectra are always non-empty compact subsets of \mathbb{C} .

2.3 Linear Operators

By $L(X)$ we denote the set of all bounded and linear operators A on the Banach space X which, equipped with point-wise addition and scalar multiplication and the usual operator norm

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|_X}{\|x\|_X} = \sup_{x \in X, \|x\|_X=1} \|Ax\|_X,$$

is a Banach space as well. Using the composition of two operators as multiplication in $L(X)$, it is also a Banach algebra with unit $I : x \mapsto x$, the identity operator on X .

By $K(X)$ we denote the set of all compact operators A on X ; that means, $T \in K(X)$ if T maps the unit ball of X to a relatively compact set in X . It is well-known that $K(X)$ is contained in $L(X)$, where it forms a closed ideal.

As usual, we say that an operator $A \in L(X)$ is *invertible*, if it is an invertible element of the Banach algebra $L(X)$. This is the case if and only if $A : X \rightarrow X$ is bijective, since, by Banach's theorem on the inverse operator (an immediate consequence of the open mapping theorem), this already implies the linearity and boundedness of the inverse operator A^{-1} .

Consequently, if A is not invertible, then $\ker A \neq \{0\}$ or $\operatorname{im} A \neq X$, or both, where, as usual,

$$\ker A = \{x \in X : Ax = 0\} \quad \text{and} \quad \operatorname{im} A = \{Ax : x \in X\}$$

denote the *kernel* (or *null space*) and the *image* (or *range*) of the operator $A \in L(X)$. As an indication of how badly injectivity and surjectivity of A are violated, one defines the two numbers

$$\alpha(A) := \dim \ker A \quad \text{and} \quad \beta(A) := \operatorname{codim}_X \operatorname{im} A \quad (2.1)$$

and calls $A \in L(X)$ a *Fredholm operator* if both numbers $\alpha(A)$ and $\beta(A)$ are finite (in which case its image is automatically closed). In that case, their difference

$$\operatorname{ind} A := \alpha(A) - \beta(A)$$

is called the *index* of A . We will call $A \in L(X)$ a *semi-Fredholm operator* if one of the two numbers $\alpha(A)$ and $\beta(A)$ is finite and if $\operatorname{im} A$ is closed.

It turns out that $A \in L(X)$ is a Fredholm operator if and only if there are operators $B, C \in L(X)$ such that

$$AB = I + T_1 \quad \text{and} \quad CA = I + T_2 \quad (2.2)$$

hold with some $T_1, T_2 \in K(X)$. The operators B and C are called right and left Fredholm regularizers of A , respectively.

By evaluating the term CAB , it becomes evident that B and C only differ by an operator in $K(X)$. Consequently, B (as well as C) is a regularizer from both sides, showing that A is Fredholm if and only if there is an operator $B \in L(X)$ such that

$$AB = I + T_1 \quad \text{and} \quad BA = I + T_2 \quad (2.3)$$

hold with some $T_1, T_2 \in K(X)$.

Obviously, (2.3) is equivalent to the invertibility of the coset $A + K(X)$ in the factor algebra $L(X)/K(X)$ where $(A + K(X))^{-1} = B + K(X)$. This fact is sometimes called Calkin's theorem, and the factor algebra $L(X)/K(X)$ is the so-called *Calkin algebra*. It is well-known that, if a coset $A + K(X)$ is invertible in the Calkin algebra, then all elements of $A + K(X)$ are Fredholm operators with the same index. For instance, all operators in $I + K(X)$ are Fredholm and have index zero.

In analogy to the spectrum $\text{spec} A$ of an operator A as an element of the Banach algebra $L(X)$, the *essential spectrum* of $A \in L(X)$ is

$$\text{spec}_{\text{ess}} A = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}.$$

By Calkin's theorem, we have that $\text{spec}_{\text{ess}} A = \text{spec}_{L(X)/K(X)}(A + K(X))$.

For a more detailed coverage of the theory of Fredholm operators, including proofs, see e.g. [53, 74, 72].

In addition to spectrum and essential spectrum, we also define, for $\varepsilon > 0$, the ε -*pseudospectrum* of $A \in L(X)$, $\text{spec}_{\varepsilon}(A)$, by

$$\text{spec}_{\varepsilon}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible or } \|(\lambda I - A)^{-1}\| \geq \varepsilon^{-1}\}.$$

We will say that a sequence $A_1, A_2, \dots \in L(X)$ converges *uniformly* (or *in the norm*) to $A \in L(X)$ and write $A_n \rightrightarrows A$ if it converges in the Banach space $L(X)$, that is $\|A_n - A\| \rightarrow 0$ as $n \rightarrow \infty$, and we will say it converges *strongly* (or *pointwise*) and write $A_n \rightarrow A$ if $\|A_n x - Ax\| \rightarrow 0$ in X for every $x \in X$.

2.4 Spaces of Sequences

We study spaces of functions $u : \mathbb{Z}^N \rightarrow X$ with $N \in \mathbb{N}$ and X an arbitrary complex Banach space. We often think of such functions as generalised sequences $u = (u(m))_{m \in \mathbb{Z}^N}$ of their function values $u(m) \in X$. Our particular focus is on the following spaces.

Definition 2.1 Let $E = \ell^p(\mathbb{Z}^N, X)$, with $1 \leq p \leq \infty$, be the set of all sequences $u = (u(m))_{m \in \mathbb{Z}^N}$ with values $u(m) \in X$, for which the following norm is finite

$$\|u\| := \begin{cases} \sqrt[p]{\sum_{m \in \mathbb{Z}^N} \|u(m)\|_X^p}, & 1 \leq p < \infty, \\ \sup_{m \in \mathbb{Z}^N} \|u(m)\|_X, & p = \infty. \end{cases}$$

Moreover, we consider the space $E = c_0(\mathbb{Z}^N, X)$, which is the closure in $\ell^\infty(\mathbb{Z}^N, X)$ of the space $c_{00}(\mathbb{Z}^N, X)$ of all sequences $u = (u(m))_{m \in \mathbb{Z}^N}$ with only finitely many nonzero entries.

Since the parameter $N \in \mathbb{N}$ is of no big importance in almost all of what follows, we will use the abbreviations $E^0(X) := c_0(\mathbb{Z}^N, X)$ and $E^p(X) := \ell^p(\mathbb{Z}^N, X)$ for $1 \leq p \leq \infty$. If there is no danger of confusion about what X is, we will even write E^0 and E^p . Many of the following statements hold for all the spaces under consideration. In this case we will simply write E , which then can be replaced by any of E^0 and E^p with $1 \leq p \leq \infty$.

Note that this setup does not limit us to functions in discrete variables. Indeed, if we put $X = L^p([0, 1]^N)$ for $p \in [1, \infty]$ then, in a natural way, we can identify elements $u \in E^p(X)$ with (equivalence classes of) scalar-valued functions f on \mathbb{R}^N via

$$(u(m))(t) = f(m + t), \quad m \in \mathbb{Z}^N, \quad t \in [0, 1]^N. \quad (2.4)$$

Indeed, via (2.4), $E^p(X)$ is identified isometrically with $L^p(\mathbb{R}^N)$, the Banach space of those Lebesgue measurable complex-valued functions f on \mathbb{R}^N , for which the norm $\|f\|_p$ is finite, where

$$\|f\|_p := \begin{cases} \sqrt[p]{\int_{\mathbb{R}^N} |f(x)|^p dx}, & 1 \leq p < \infty, \\ \text{ess sup}_{x \in \mathbb{R}^N} |f(x)|, & p = \infty. \end{cases}$$

2.5 An Approximate Identity

Let E be one of the sequence spaces introduced in the previous section. A first important class of operators on E is the following.

Definition 2.2 Consider a set $U \subset \mathbb{Z}^N$. We define P_U as the operator that acts on E by

$$(P_U u)(m) := \begin{cases} u(m) & \text{if } m \in U, \\ 0 & \text{if } m \notin U \end{cases}$$

with $m \in \mathbb{Z}^N$. Clearly, P_U is a projector. We will refer to its complementary projector $I - P_U$ by Q_U .

Typical examples of projectors P_U we have to deal with are of the form $P_n := P_{U_n}$ with

$$U_n = \{m \in \mathbb{Z}^N : |m| \leq n\} = \{-n, \dots, n\}^N$$

with some $n \in \mathbb{N}$. Again, put $Q_n := I - P_n$.

In connection with approximation methods, but also for the classification of operators and notions of convergence, we will look at a sequence of such projectors that is increasing in an appropriate sense. We will use the sequence

$$\mathcal{P} := (P_1, P_2, P_3, \dots) \quad (2.5)$$

with P_1, P_2, P_3, \dots as in Definition 2.2. \mathcal{P} is an *approximate identity* in the terminology of [143, 39]; precisely, it is subject to the constraints

- (i) $_{\mathcal{P}}$ $\sup_n \|P_n u\| = \|u\|$ for all $u \in E$;
- (ii) $_{\mathcal{P}}$ for every $m \in \mathbb{N}$ there exists $N(m) \geq m$ such that

$$P_n P_m = P_m = P_m P_n, \quad n \geq N(m).$$

In [143] a bounded sequence \mathcal{P} satisfying (ii) $_{\mathcal{P}}$ is called an *increasing approximate projection* (note that the operators P_n do not need to be projection operators, i.e. subject to $P_n^2 = P_n$, themselves) and an increasing approximate projection satisfying (i) $_{\mathcal{P}}$ (or (i) $_{\mathcal{P}}$ with the ‘=’ replaced by a ‘ \geq ’) is called an *approximate identity*. Thus \mathcal{P} is an approximate identity in the terminology of [143].

Besides our operators P_n as introduced in Definition 2.2 on our sequence spaces E , there is clearly a much greater variety of operator sequences P_1, P_2, \dots on these or other Banach spaces E that meet conditions (i) $_{\mathcal{P}}$ and (ii) $_{\mathcal{P}}$. We briefly give some examples here and refer to [143, 39] for a more general theory on approximate identities before we go back to (2.5) with P_1, P_2, \dots from Definition 2.2.

Example 2.3 Let $E = \ell^\infty(\mathbb{Z}, \mathbb{C})$, the Banach space of bounded complex-valued sequences $u = (u(m))_{m \in \mathbb{Z}}$, with norm $\|u\| = \sup_m |u(m)|$. Define, for $u \in E$ and $n \in \mathbb{N}_0$, $P_n u \in E$ by the two conditions that $(P_n u)(m) = u(m)$ for $|m| \leq (3^n - 1)/2$ and that $(P_n u)(m + 3^n) = (P_n u)(m)$ for $m \in \mathbb{Z}$. Then $\mathcal{P} = (P_n)$ satisfies (i) $_{\mathcal{P}}$ and (ii) $_{\mathcal{P}}$ with $N(m) = m$, so that P_n is a projection operator for each n . \square

Example 2.4 Let $E = BC(\mathbb{R}^N)$, the Banach space of bounded continuous complex-valued functions on \mathbb{R}^N with norm $\|f\| = \sup_{x \in \mathbb{R}^N} |f(x)|$. Choose $\chi \in BC(\mathbb{R})$ with $\|\chi\| = 1$, $\chi(x) = 0$ for $x \leq 0$ and $\chi(x) = 1$ for $x \geq 1$. Define, for $n \in \mathbb{N}$ and $f \in E$,

$$(P_n f)(x) = \chi(n+1 - |x|)f(x), \quad x \in \mathbb{R}^N.$$

Then $\mathcal{P} = (P_n)$ satisfies $(i)_{\mathcal{P}}$ and $(ii)_{\mathcal{P}}$ with $N(m) = m+1$. In this case $\|Q_n\| = \|1 - \chi\|$. \square

Example 2.5 Let $E = C[0, 1]$ with $\|f\| = \sup_{x \in [0, 1]} |f(x)|$ and let $P_n f$ denote the piecewise linear function which interpolates f at $j/2^n$, $j = 0, 1, \dots, 2^n$. Then $\mathcal{P} = (P_n)$ satisfies $(i)_{\mathcal{P}}$ and $(ii)_{\mathcal{P}}$ with $N(m) = m$. \square

2.6 Different Topologies on E

Let E be one of the sequence spaces $E^p(X)$ introduced above with $p \in \{0\} \cup [1, \infty]$ and a complex Banach space X .

2.6.1 The Norm Topology

Equipped with the corresponding norm $\|\cdot\|$ from Definition 2.1, E is a Banach space. We will write $u_n \rightarrow u$ for convergence of a sequence $u_1, u_2, \dots \in E$ in this norm, i.e. $\|u_n - u\| \rightarrow 0$ to an element $u \in E$. The topology associated with $(E, \|\cdot\|)$ will be called the *norm topology*.

Let E_{00} denote the linear subspace of E that consists of all sequences u with only finitely many nonzero entries, that is

$$E_{00} = \bigcup_{n \in \mathbb{N}} \text{im } P_n,$$

and let E_0 be the closure of E_{00} in $(E, \|\cdot\|)$.

Lemma 2.6 *It holds that $E_0 = \{u \in E : Q_n u \rightarrow 0 \text{ as } n \rightarrow \infty\}$ so that $E = E_0$ iff $P_n \rightarrow I$ strongly as $n \rightarrow \infty$, that is iff $p \neq \infty$.*

Proof. The claim follows from $(i)_{\mathcal{P}}$ and $(ii)_{\mathcal{P}}$ (e.g. [143]) where in our situation one even has $\|Q_n u\| = \text{dist}(u, \text{im } P_n)$ for every $n \in \mathbb{N}$ and $u \in E$, by Definitions 2.1 and 2.2. \blacksquare

We will also be concerned with convergence in weaker topologies on E , defined in terms of semi-norms that are related to the approximate projection \mathcal{P} .

2.6.2 The Local Topology

For every $n \in \mathbb{N}$ and $u \in E$, put

$$|u|_n := \|P_n u\|.$$

Then $\{|\cdot|_n : n \in \mathbb{N}\}$ is a countable family of seminorms on E which is separating points since, by conditions $(i)_{\mathcal{P}}$ and $(ii)_{\mathcal{P}}$ above,

$$\|u\| = \sup_n \|P_n u\| = \sup_n |u|_n = \lim_{n \rightarrow \infty} |u|_n.$$

We call the metrisable topology generated by this family of semi-norms the *local topology*. Equipped with this topology, in which case we write (E, loc) , E is a separated locally convex topological vector space (TVS). By definition, a sequence (u_n) of E converges to $u \in E$ in the local topology if and only if $|u_n - u|_m \rightarrow 0$, that is $P_m u_n \rightarrow P_m u$, as $n \rightarrow \infty$ for all $m \in \mathbb{N}$, i.e. it converges pointwise

$$u_n(k) \rightarrow u(k) \quad \text{as } n \rightarrow \infty, \quad \text{for all } k \in \mathbb{Z}^N.$$

We will also be interested in a third topology on E , intermediate between the local and norm topologies.

2.6.3 The Strict Topology

Given a positive null-sequence $a : \mathbb{N} \rightarrow (0, \infty)$ and $u \in E$, define

$$|u|_a := \sup_n a(n) |u|_n.$$

Then $\{|\cdot|_a : a \text{ is a positive null-sequence}\}$ is another separating family of seminorms on E and generates another separated locally convex topology. By analogy with [25], we term it the *strict topology* and write (E, s) for E equipped with the strict topology. For $(u_n) \subset E$, $u \in E$, we write $u_n \xrightarrow{s} u$ if u_n converges to u in (E, s) , i.e. if $|u_n - u|_a \rightarrow 0$ as $n \rightarrow \infty$ for every positive null-sequence a .

The strict topology (called the β topology in [46]) has been extensively studied in [25, 46, 39]. Various properties of the β /strict topology are shown in [46, Theorem 2.1], in large part adapting arguments from [25]. The properties that we need for our arguments are summarised in the next lemma, which is a special case of [39, Lemma 2.11] (note that $E = \hat{E}$ in the notations of [39] if $p \neq 0$).

As usual we will call a set S in a TVS E *bounded* if it is absorbed by every neighbourhood of zero and *totally bounded* if, for every neighbourhood of zero, U , there exists a finite set $\{a_1, \dots, a_N\} \subset E$ such that $\cup_{1 \leq j \leq N} (a_j + U)$ contains S . Every totally bounded set is bounded [151].

Lemma 2.7 *Let $E = E^p$ with $p \in [1, \infty]$.*

- (i) *In E the bounded sets in the strict topology and the norm topology are the same.*
- (ii) *On every norm-bounded subset of E the strict topology coincides with the local topology.*
- (iii) *A sequence $(u_n) \subset E$ is convergent in the strict topology iff it is convergent in the local topology and is bounded in the norm topology, so that*

$$u_n \xrightarrow{s} u \quad \Leftrightarrow \quad \sup_n \|u_n\| < \infty \text{ and } P_m u_n \rightarrow P_m u \text{ as } n \rightarrow \infty, \text{ for all } m. \quad (2.6)$$

- (iv) *A norm-bounded subset of E is closed in the strict topology iff it is sequentially closed.*
- (v) *A sequence in E is Cauchy in the strict topology iff it is Cauchy in the local topology and bounded in the norm topology.*
- (vi) *Let $S \subset E$. Then the following statements are equivalent:*
 - (a) *S is totally bounded in the strict topology.*
 - (b) *S is norm-bounded and totally bounded in the local topology.*
 - (c) *Every sequence in S has a subsequence that is Cauchy in the strict topology.*

Lemma 2.8 (i) *On E the local topology is strictly coarser than the strict topology which is strictly coarser than the norm topology.*

(ii) *(E, loc) is metrisable but not complete, while (E, s) is complete if $p \neq 0$ but non-metrisable.*

Proof. (i) Take $u_1, u_2, \dots \in E$ such that $\|Q_n u_n\| = 1$ for all n . Clearly $Q_n u_n \not\rightarrow 0$, but it follows from (2.6) that $Q_n u_n \xrightarrow{s} 0$ as $n \rightarrow \infty$. Thus the strict and norm topologies are distinct. To see that the local and strict topologies are distinct, note that $nQ_n u_n$ converges to zero in the local topology but $\|nQ_n u_n\| = n \rightarrow \infty$ so that, by (2.6), $nQ_n u_n \not\xrightarrow{s} 0$.

(ii) Since the local topology is generated by the countable family of semi-norms $|\cdot|_n$ it is metrisable. If (E, loc) were complete it would be a Fréchet space and it would follow from the open mapping theorem [157] applied to the identity operator that the local and norm topologies coincide – which they don't, by (i).

Let \overline{E}_{loc} and \overline{E}_s denote the completion of E in the local and strict topology, respectively. Then $\overline{E}_s \subset \overline{E}_{loc}$ by part (i). Suppose $\overline{E}_s \neq E$. Then there exists $u \in \overline{E}_s$ with $|u|_n \rightarrow \infty$ as $n \rightarrow \infty$ (note that each $|\cdot|_n$ extends continuously to $\overline{E}_{loc} \supset \overline{E}_s$). Let $a(n) := 2 \min(1, 1/|u|_n)$ be a positive null-sequence. Then $v \in \overline{E}_s$ and $|u - v|_a < 1$ imply that $|v|_n > |u|_n/2$ for all sufficiently large n , so that $\{v \in E : |u - v|_a < 1\} = \emptyset$. This is a contradiction, for E is dense in its completion.

If (E, s) were metrisable (and complete) then the above argument using the open mapping theorem in Fréchet spaces would contradict part (i). ■

By definition, E_0 is the completion of E_{00} in the norm topology and we have seen in Lemma 2.6 that $Q_n u \rightarrow 0$ iff $u \in E_0$. The next lemma states corresponding results for the strict topology.

Lemma 2.9 *For every $u \in E$, it holds that $Q_n u \xrightarrow{s} 0$ as $n \rightarrow \infty$. Further, if $p \neq 0$, the completion of E_{00} in the strict topology is E , so that (E, s) is sequentially complete.*

Proof. By Lemma 2.7 (iii), we have $P_n u \xrightarrow{s} u$ for every $u \in E$ since $\|P_n u\|$ is bounded by $\|u\|$. Since $P_n u \in E_{00}$ for every n , the completion of E_{00} contains E ; in fact it coincides with E since (E, s) is complete by Lemma 2.8 (ii). Since (E, s) is complete and sequentially closed it is sequentially complete. ■

As usual, we will call a subset S of a topological space *compact* if every open cover of S has a finite subcover, *relatively compact* if its closure is compact, and we call it *relatively sequentially compact* if every sequence in S has a subsequence converging to a point in the topological space.

Lemma 2.10 *Let $S \subset E$. Then S is compact in (E, s) iff it is sequentially compact. Further, if $p \neq 0$, the following are equivalent:*

- (a) *S is relatively compact in the strict topology.*
- (b) *S is relatively sequentially compact in the strict topology.*
- (c) *S is totally bounded in the strict topology.*
- (d) *S is norm-bounded and $P_n(S)$ is relatively compact in the norm topology for each n .*

If $p = 0$ then $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d)$ holds.

Proof. To show that compactness (relative compactness) of S is equivalent to sequential compactness (relative sequential compactness) it is enough to show this in the strict topology restricted to \bar{S} , the closure of S in (E, s) . But, if S is relatively sequentially compact or relatively compact then it is bounded and so \bar{S} is bounded. But, by (ii) of Lemma 2.7 (if $p = 0$ note that $S \subset E = E^0 \subset E^\infty$ and apply Lemma 2.7 with $p = \infty$), the strict topology coincides with the metrisable local topology on bounded sets, and in metric spaces compactness and sequential compactness coincide. Thus the first statement of the theorem holds and also (a) \Leftrightarrow (b). That (b) implies (c), and the converse if (E, s) is sequentially complete (i.e. if $p \neq 0$), is immediate from (vi) of Lemma 2.7. If (c) holds then, also by (vi) of Lemma 2.7, S is norm-bounded and every sequence in S has a subsequence that is Cauchy in the strict topology. Since P_n is continuous from (E, s) to $(E, \|\cdot\|)$ and since $(E, \|\cdot\|)$ is complete, this implies that $P_n(S)$ is relatively compact in the norm topology. Finally, suppose (d) holds and take an arbitrary bounded sequence $(u_n) \subset E$. Choose a subsequence $(u_n^{(1)})$ such that $P_1 u_n^{(1)}$ norm-converges as $n \rightarrow \infty$. From $(u_n^{(1)})$ choose a subsequence $(u_n^{(2)})$ such that $P_2 u_n^{(2)}$ norm-converges, and so on. Then (v_n) , with $v_n := u_n^{(n)}$, which is bounded and Cauchy in the local topology is Cauchy in the strict topology by Lemma 2.7 (iv). Thus every sequence in S has a subsequence that is Cauchy in the strict topology, so that, by Lemma 2.7 (vi), (c) holds. ■

As the following corollary of the above lemma already indicates, many of the results we obtain in the text will simplify and become more complete in the case that $P_n \in K(E)$ for all n – that is when the Banach space X in our setting $E = E^p(X)$ is finite-dimensional.

Corollary 2.11 *If $P_n \in K(E)$ for all n , then a set $S \subset E$ is relatively compact in the strict topology iff it is norm-bounded.*

2.7 Comments and References

The idea to study ℓ^p sequences with values in a Banach space X and to identify $L^p(\mathbb{R}^N)$ with such a space has a long history. It can be found in [95], [22], [81], [141] and [143], to give some of the more recent references.

Approximate identities are introduced as special approximate projections in [143]. Their applications go far beyond the projectors presented here. For more general studies and examples, see e.g. [81], [143] and [39].

The study of the strict topology was initiated by Buck [25] and later extended in many directions. The results presented here go back to Chandler-Wilde and Zhang [46].

Chapter 3

Classes of Operators

We continue to suppose that E is one of our sequence spaces E^p introduced in Section 2.4 with $p \in \{0\} \cup [1, \infty]$. We have already introduced $L(E)$ and $K(E)$, the sets of linear operators that are, respectively, bounded and compact on the Banach space $(E, \|\cdot\|)$. Now we first look at operators with continuity and compactness properties on the TVS (E, s) introduced in Section 2.6 or between (E, s) and $(E, \|\cdot\|)$. We then continue by specifying the classes of operators that we are studying in the chapters that follow.

3.1 Continuous Operators on (E, s)

It follows from (i) of Lemma 2.7 that the linear operators on E that are bounded on (E, s) (i.e. map bounded sets to bounded sets) are precisely those that are bounded on $(E, \|\cdot\|)$, namely the members of $L(E)$.

Now let $S(E)$ denote the set of those linear operators that are sequentially continuous on (E, s) . Thus $A \in S(E)$ if and only if, for every sequence $(u_n) \subset E$ and $u \in E$,

$$u_n \xrightarrow{s} u \quad \implies \quad Au_n \xrightarrow{s} Au. \quad (3.1)$$

Lemma 3.1 *It holds that $A \in S(E)$ iff A is continuous on (E, s) .*

Proof. From standard properties of TVS's (e.g. [157, Theorems A6 and 1.30]) and Lemma 2.8 it follows that every continuous linear operator A on (E, s) is sequentially continuous, i.e. $A \in S(E) \subset L(E)$. To see the reverse implication in case $p \neq 0$, put $E_n := \text{im } Q_n$ for all $n \in \mathbb{N}$, so that Assumption A' of [46] holds. The claim then follows from [46, Theorem 3.7]. For $p = 0$, the claim follows from the case $p = \infty$ and Lemma 3.15 below. ■

In analogy to $S(E)$, let $SN(E)$ denote the set of those linear operators that are sequentially continuous from (E, s) to $(E, \|\cdot\|)$, so that $A \in SN(E)$ iff

$$u_n \xrightarrow{s} u \quad \implies \quad Au_n \rightarrow Au. \quad (3.2)$$

We remark that the operators in $S(E)$ and $SN(E)$ are precisely those termed s -continuous and sn -continuous, respectively, in [9].

It clearly holds that

$$SN(E) \subset S(E) \subset L(E). \quad (3.3)$$

As Lemmas 3.2 and 3.3 below clarify, in general $SN(E)$ is a strict subset of $S(E)$. In Example 3.12 below we will see that also the inclusion $S(E) \subset L(E)$ is proper and indeed that $A \in L(E)$ may be even compact on $(E, \|\cdot\|)$ but not sequentially continuous on (E, s) .

The following lemmas provide alternative characterisations of the classes $SN(E)$ and $S(E)$ and shed some light on the relationship with $K(E)$. In particular, we show that if A is compact on $(E, \|\cdot\|)$ and sequentially continuous on (E, s) then $A \in SN(E)$.

Lemma 3.2 $A \in SN(E)$ iff $A \in L(E)$ and $\|AQ_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Suppose $A \in SN(E)$. Then $A \in L(E)$. To see that also $\|AQ_n\| \rightarrow 0$ as $n \rightarrow \infty$, suppose that this does not hold. Then there is a bounded sequence $(u_n) \subset E$ such that $AQ_n u_n \not\rightarrow 0$. But this is impossible as $Q_n u_n \xrightarrow{s} 0$, and hence $\|AQ_n u_n\| \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction.

For the reverse implication, take an arbitrary sequence $(u_n) \subset E$ with $u_n \xrightarrow{s} 0$ as $n \rightarrow \infty$. Then $\|u_n\|$ is bounded and $\|P_m u_n\| \rightarrow 0$ as $n \rightarrow \infty$ for every m . Now, for every m and n ,

$$\begin{aligned} \|Au_n\| &\leq \|AP_m u_n\| + \|AQ_m u_n\| \\ &\leq \|A\| \|P_m u_n\| + \|AQ_m\| \sup_n \|u_n\| \end{aligned}$$

holds, where $\|AQ_m\|$ can be made as small as desired by choosing m large enough, and $\|P_m u_n\|$ tends to zero as $n \rightarrow \infty$. ■

Lemma 3.3 $A \in S(E)$ iff $A \in L(E)$ and $P_m A \in SN(E)$ for every m .

Proof. If $A \in S(E)$ then $A \in L(E)$. The rest trivially follows from

$$Au_n \xrightarrow{s} 0 \text{ as } n \rightarrow \infty \quad \iff \quad \|P_m A u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \quad \forall m$$

for every bounded operator A and every bounded sequence $(u_n) \subset E$. ■

Corollary 3.4 $A \in S(E)$ iff $A \in L(E)$ and $\|P_m A Q_n\| \rightarrow 0$ as $n \rightarrow \infty$, $\forall m \in \mathbb{N}$.

In Lemma 3.1 we have seen that continuity and sequential continuity for operators $(E, s) \rightarrow (E, s)$ are the same. Here is Lemma 3.9 from [39] – an analogous result for mappings $(E, s) \rightarrow (E, \|\cdot\|)$.

Lemma 3.5 *The following are equivalent for a linear operator A on E .*

- (a) $A \in SN(E)$.
- (b) $A \in L(E)$ and there is a neighbourhood of zero, U , in (E, s) , for which $A(U)$ is norm-bounded, in fact for which $\sup_{u \in U} \|A Q_n u\| \rightarrow 0$ as $n \rightarrow \infty$.
- (c) A is a continuous mapping from (E, s) to $(E, \|\cdot\|)$.

Having these characterisations of $S(E)$ and $SN(E)$, we now look into their interrelations with $K(E)$.

Lemma 3.6 $S(E) \cap K(E) \subseteq SN(E)$ with equality if and only if $P_n \in K(E)$ for all n , that is iff $\dim X < \infty$ where $E = E^p(X)$.

Proof. Suppose $A \in S(E) \cap K(E)$. Take an arbitrary sequence $(u_n) \subset E$ with $u_n \xrightarrow{s} 0$ as $n \rightarrow \infty$. From $A \in S(E)$ we conclude that $Au_n \xrightarrow{s} 0$ as $n \rightarrow \infty$. Since $\{u_n\}$ is bounded and A is compact, we know that $\{Au_n\}$ is relatively compact; so every subsequence of (Au_n) has a norm-convergent subsequence, where the latter can only have limit 0 since $Au_n \xrightarrow{s} 0$ as $n \rightarrow \infty$. Of course, this property ensures that Au_n itself norm-converges to 0.

To see when equality holds consider that, for all m , $P_m Q_n = 0$ for all sufficiently large n . Thus, by Lemma 3.2, $P_m \in SN(E)$ for all m . So clearly $SN(E) \not\subseteq K(E)$ if P_m is not compact for some m . If P_m is compact for all m and $A \in SN(E)$ then, by Lemma 3.2 again, A is the norm limit of AP_m as $m \rightarrow \infty$, with AP_m compact for all m , so that A is compact. Thus equality holds iff $P_m \in K(E)$ for all m . ■

3.2 Compact Operators and Generalisations

3.2.1 Compact Operators on $(E, \|\cdot\|)$ and Generalisations

One crucial property of compact operators $K \in K(E)$ is that, since pointwise convergence is uniform on compact sets, they turn strong convergence into norm

convergence if they are applied to the convergent sequence from the right; that is, $A_n \rightarrow A$ implies $A_n K \rightrightarrows AK$ as $n \rightarrow \infty$.

If $p \in \{0\} \cup [1, \infty)$ (also recall Lemma 2.6), we have that $Q_n u \rightarrow 0$ for all $u \in E_0 = E = E^p$ and therefore

$$\|Q_n K\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (3.4)$$

for all $K \in K(E)$. If also $p \neq 1$, i.e. if $p \in \{0\} \cup (1, \infty)$, then we have for the adjoint operators that $Q_n^* \rightarrow 0$ strongly on the dual space $E^* \cong E^q(X^*)$ of $E = E^p(X)$ with $1/p + 1/q = 1$, so that $\|K Q_n\| = \|(K Q_n)^*\| = \|Q_n^* K^*\|$ tends to zero, i.e.

$$\|K Q_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad (3.5)$$

for all $K \in K(E)$. Note that in the latter case $K(E) \subset SN(E)$ by Lemma 3.2.

We now change perspective and modify the set $K(E)$ so that both (3.4) and (3.5) hold for all K in the new set, for **all** spaces $E = E^p$. To this end let $K(E, \mathcal{P})$ denote the set of all $K \in L(E)$ for which (3.4) and (3.5) hold. Moreover, let $L(E, \mathcal{P})$ refer to the set of all bounded linear operators A on E such that AK and KA are both in $K(E, \mathcal{P})$ whenever $K \in K(E, \mathcal{P})$.

Both $K(E, \mathcal{P})$ and $L(E, \mathcal{P})$ are Banach subalgebras of $L(E)$, and $K(E, \mathcal{P})$ is an ideal (two-sided, closed) in $L(E, \mathcal{P})$. By definition, $L(E, \mathcal{P})$ is the largest subalgebra of $L(E)$ with that property – the so-called idealiser of $K(E, \mathcal{P})$ in $L(E)$. It is shown in [143, Theorem 1.1.9] that $L(E, \mathcal{P})$ is *inverse closed*; that is, if $A \in L(E, \mathcal{P})$ is invertible as an element of $L(E)$ (i.e. a bijection $E \rightarrow E$) then $A^{-1} \in L(E, \mathcal{P})$.

Lemma 3.7 *An operator $A \in L(E)$ is in $L(E, \mathcal{P})$ iff, for every $m \in \mathbb{N}$,*

$$\|P_m A Q_n\| \rightarrow 0 \quad \text{and} \quad \|Q_n A P_m\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (3.6)$$

Proof. First suppose $A \in L(E, \mathcal{P})$ and take an $m \in \mathbb{N}$. From $P_m \in K(E, \mathcal{P})$ we get that $P_m A, A P_m \in K(E, \mathcal{P})$ which shows (3.6).

Now suppose that (3.6) is true and take an arbitrary $K \in K(E, \mathcal{P})$. To see that $AK \in K(E, \mathcal{P})$, note that $(AK)Q_n = A(KQ_n) \rightrightarrows 0$ as $n \rightarrow \infty$, and that,

$$\|Q_n(AK)\| \leq \|Q_n A P_m\| \cdot \|K\| + \|Q_n A\| \cdot \|Q_m K\|$$

holds for every $m \in \mathbb{N}$, where, by (3.6), the first term tends to zero as $n \rightarrow \infty$, and $\|Q_m K\|$ can be made as small as desired by choosing m large enough. By a symmetric argument, one shows that also $KA \in K(E, \mathcal{P})$, and hence, $A \in L(E, \mathcal{P})$. ■

If $E = E^p(X)$ is a Hilbert space, i.e. if $p = 2$ and X is Hilbert space, an operator $A \in L(E)$ is called a *quasidiagonal operator* with respect to \mathcal{P} (as introduced by Halmos [79]) if $[P_n, A] \rightrightarrows 0$ holds as $n \rightarrow \infty$, where we let $[A, B]$ refer to the *commutator* of two operators $A, B \in L(E)$; that is $[A, B] = AB - BA$. It is readily checked that the class of quasidiagonal operators is contained in $L(E, \mathcal{P})$, even if we generalise that definition to $p \in \{0\} \cup [1, \infty]$ and to arbitrary Banach spaces X . Indeed, if A is quasidiagonal and $n \geq m$, then

$$P_m A Q_n = P_m A - P_m A P_n = P_m (P_n A - A P_n) \rightrightarrows 0 \quad \text{as} \quad n \rightarrow \infty.$$

By a symmetric argument we see that A also has the second property in (3.6), and consequently, $A \in L(E, \mathcal{P})$. The reverse implication, namely that $A \in L(E, \mathcal{P})$ implies $[P_n, A] \rightrightarrows 0$, is clearly false but there is the following description of the class $L(E, \mathcal{P})$ in terms of the commutator $[P_n, A]$.

Lemma 3.8 *$A \in L(E)$ is contained in $L(E, \mathcal{P})$ if and only if, for every $n \in \mathbb{N}$,*

$$[P_n, A] \in K(E, \mathcal{P}).$$

Proof. Clearly, if $A \in L(E, \mathcal{P})$, then $P_n A, A P_n \in K(E, \mathcal{P})$, whence also the commutator $[P_n, A] = P_n A - A P_n$ is in $K(E, \mathcal{P})$ for every $n \in \mathbb{N}$.

For the reverse direction, note that for every fixed $m \in \mathbb{N}$,

$$P_m A Q_n = [P_m, A] Q_n + A P_m Q_n \rightrightarrows 0 \quad \text{as} \quad n \rightarrow \infty$$

since $[P_m, A] \in K(E, \mathcal{P})$ and $P_m Q_n = 0$ for all $n \geq m$. Analogously, we prove the second property in (3.6), showing that $A \in L(E, \mathcal{P})$. ■

The definition of $K(E, \mathcal{P})$ and the characterisation of $L(E, \mathcal{P})$ by Lemma 3.7 bear a close resemblance to the characterisations of $SN(E)$ and $S(E)$ in Lemma 3.2 and Corollary 3.4. Roughly speaking, $L(E, \mathcal{P})$ and $K(E, \mathcal{P})$ are two-sided versions of $S(E)$ and $SN(E)$, respectively. One clearly has

$$L(E, \mathcal{P}) \subset S(E) \quad \text{and} \quad K(E, \mathcal{P}) \subset SN(E),$$

and in the case that E is a Hilbert space (i.e. when $E = E^2(X)$ with a Hilbert space X) and each P_n is self-adjoint, it holds that $A \in L(E, \mathcal{P})$, resp. $\in K(E, \mathcal{P})$, iff A and A^* both are in $S(E)$, resp. $SN(E)$, where A^* denotes the adjoint of A .

Another way to look at this is that, very similar to the definition of $L(E, \mathcal{P})$ as the idealiser of $K(E, \mathcal{P})$, an operator $A \in L(E)$ is in $S(E)$ iff $KA \in SN(E)$ for all $K \in SN(E)$; that is, $S(E)$ is the left-idealiser of $SN(E)$ in $L(E)$.

The characterisation of $L(E, \mathcal{P})$ by Lemma 3.7 also yields the following interesting result:

Lemma 3.9 *For an operator $K \in L(E, \mathcal{P})$, either both or neither of the two properties (3.4) and (3.5) hold, so that $L(E, \mathcal{P}) \cap SN(E) = K(E, \mathcal{P})$.*

Proof. Suppose $K \in L(E, \mathcal{P})$ and (3.5) holds. Then for all $m, n \in \mathbb{N}$,

$$\|Q_n K\| \leq \|Q_n K P_m\| + \|Q_n K Q_m\| \leq \|Q_n K P_m\| + \|K Q_m\|$$

holds, where $\|K Q_m\|$ can be made as small as desired by choosing m large enough, and $\|Q_n K P_m\|$ tends to zero as $n \rightarrow \infty$. Consequently, also property (3.4) holds. By a symmetric argument we see that property (3.4) implies (3.5) if $K \in L(E, \mathcal{P})$. ■

In analogy to Lemma 3.6 we have the following result.

Lemma 3.10 *$L(E, \mathcal{P}) \cap K(E) \subseteq K(E, \mathcal{P})$ with equality if and only if $P_n \in K(E)$ for all n , that is iff $\dim X < \infty$.*

Proof. From Corollary 3.4 and Lemma 3.7 we know that $L(E, \mathcal{P}) \subseteq S(E)$. Consequently,

$$\begin{aligned} L(E, \mathcal{P}) \cap K(E) &\subseteq L(E, \mathcal{P}) \cap S(E) \cap K(E) \\ &\subseteq L(E, \mathcal{P}) \cap SN(E) = K(E, \mathcal{P}), \end{aligned}$$

where we used Lemmas 3.6 and 3.9 for the last two steps. Moreover, if $P_n \in K(E)$ for all n and $K \in K(E, \mathcal{P})$ then $P_n K \in K(E)$ for all n and $K = \lim P_n K \in K(E)$. If $P_n \notin K(E)$ for some n then P_n is contained in the difference of the two sets under consideration. ■

The above lemma has the following refinement in the case when $E = E^p(X)$ with $p \in \{0\} \cup (1, \infty)$.

Lemma 3.11 (i) *If $p \in \{0\} \cup (1, \infty)$ then $K(E) \subset K(E, \mathcal{P})$.*

(ii) *If X is finite-dimensional then $K(E, \mathcal{P}) \subset K(E)$.*

(iii) *If both hold, $p \in \{0\} \cup (1, \infty)$ and $\dim X < \infty$, then $K(E) = K(E, \mathcal{P})$ and $L(E) = L(E, \mathcal{P})$.*

Proof. (i) We have already seen that (3.4) and (3.5) hold for all $K \in K(E)$ if $p \in \{0\} \cup (1, \infty)$.

(ii) The inclusion $K(E, \mathcal{P}) \subset K(E)$ if $\mathcal{P} \subset K(E)$ follows from Lemma 3.10.

(iii) The equality of $K(E, \mathcal{P})$ and $K(E)$ follows from (i) and (ii), and the equality $L(E) = L(E, \mathcal{P})$ is a consequence of the definition of $L(E, \mathcal{P})$. ■

The relation between $K(E, \mathcal{P})$, $K(E)$, $L(E, \mathcal{P})$ and $L(E)$ from Lemmas 3.10 and 3.11 will be visualised in a Venn diagram in Figure 3.1 below. There is a further Venn diagram in this section, Figure 3.2, which shows the relation between these and other operator classes still to be discussed in this section.

Before we come to these Venn diagrams, we first give some basic examples of operators which are in $L(E)$ but not in $L(E, \mathcal{P})$. In all these examples, the first condition in (3.6) is violated so that the operator is not even in $S(E)$. Note that some of these operators are compact and some are not. We will include these operators in the Venn diagrams in Figure 3.1. For simplicity, we restrict ourselves to the case $N = 1$.

Example 3.12 a) Our first example consists of an operator on $E^1(X)$,

$$A : (u_i) \mapsto \left(\dots, 0, 0, \sum_{i=-\infty}^{\infty} u_i, 0, 0, \dots \right),$$

where the sum is in the 0-th component.

Moreover, we consider A 's compact friend \tilde{A} on $E^1(X)$ with

$$\tilde{A} : (u_i) \mapsto \left(\dots, 0, 0, \sum_{i=-\infty}^{\infty} f(u_i) a, 0, 0, \dots \right)$$

where $f \in X^*$ and $a \in X$ are fixed non-zero elements. Note that, unlike A , the operator \tilde{A} is compact, independently of $\dim X$.

b) Our second example is the operator $B : u \mapsto v$ on $L^p(\mathbb{R}) \cong \ell^p(\mathbb{Z}, L^p([0, 1]))$ with $p \in [1, \infty]$, where

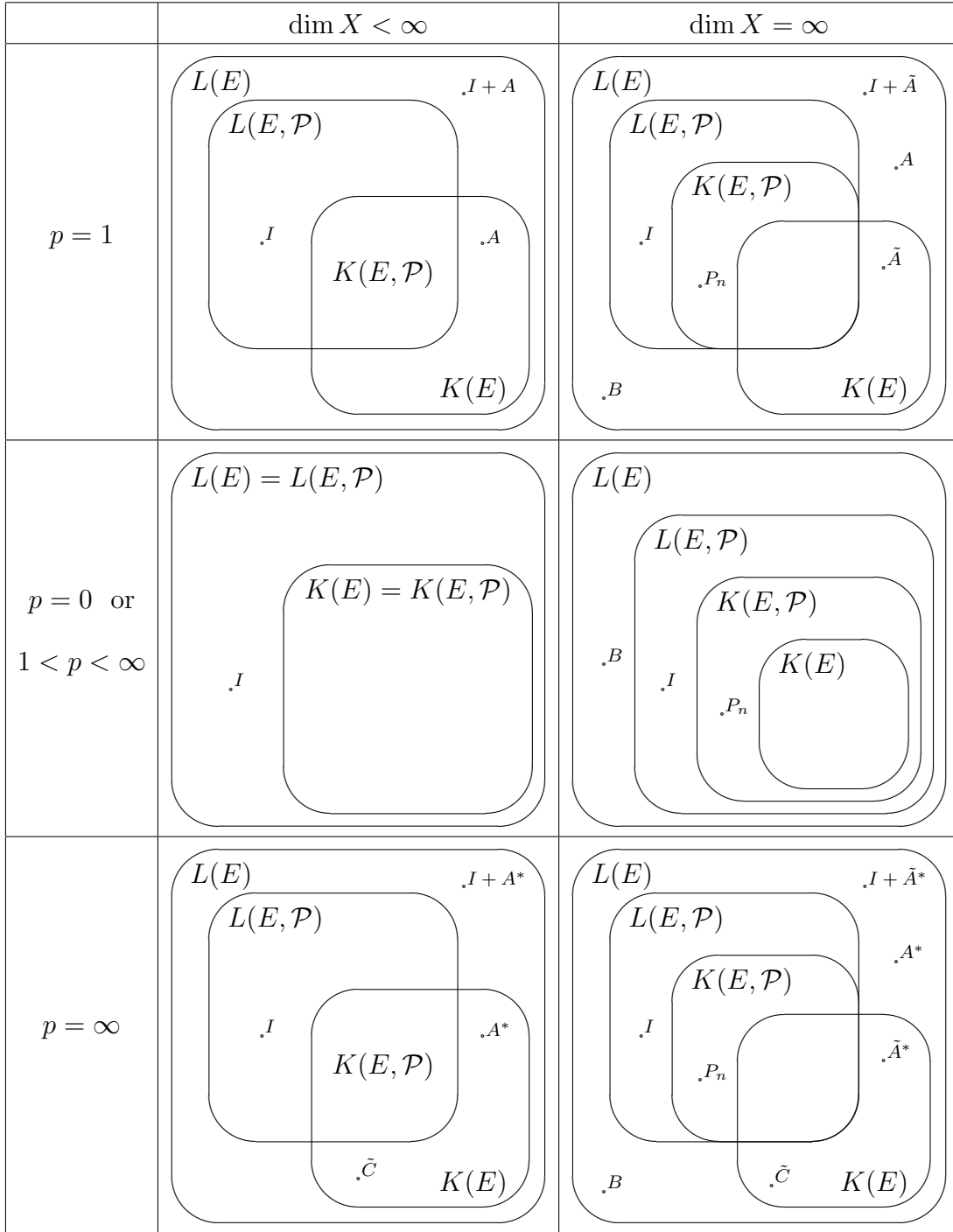
$$v(x) = \begin{cases} u(x+k), & x \in \left(1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}\right), \quad k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

c) Our last example is a compact operator C on $E = \ell^\infty(\mathbb{Z}, \mathbb{C})$ that is constructed as follows. Let c_+ denote the set of those $u \in E$ for which $\lim_{m \rightarrow +\infty} u(m)$ exists. By the Hahn-Banach theorem, a bounded linear functional $\ell_+ : E \rightarrow \mathbb{C}$ exists such that $\ell_+(u) = \lim_{m \rightarrow +\infty} u(m)$ for all $u \in c_+$. Define $C : E \rightarrow E$ by $Cu = \ell_+(u)v$, $u \in E$, where $v \in E$ is non-zero and fixed. Then the range of C is one-dimensional so that $C \in K(E) \subset L(E)$. However, defining $u = (\dots, 1, 1, 1, \dots)$ and $u_n = Q_n u$, we get that $u_n \xrightarrow{s} 0$ as $n \rightarrow \infty$ but $Cu_n = 1$ for all n . Thus $C \notin S(E)$.

The same idea can be carried over to $E = \ell^\infty(\mathbb{Z}, X)$ with a Banach space X by putting

$$\tilde{C}u = C(\dots, f(u(-1)), f(u(0)), f(u(1)), \dots), \quad u \in E$$

with a fixed non-zero functional $f \in X^*$. \square

Figure 3.1: Venn diagrams of $L(E)$, $L(E, \mathcal{P})$, $K(E, \mathcal{P})$ and $K(E)$ depending on $E = E^p(X)$.

3.2.2 Restrict and Extend Operators to and from E_0

Recall that E_0 is the closure of $E_{00} = \cup_n \text{im } P_n$ and is characterised in Lemma 2.6. It holds that $E_0 = E$ in all cases $E = E^p$ with $p \neq \infty$. But because a considerable part of our investigations takes place in $E = E^\infty$, where $E_0 = E^0$ is a proper subspace of E , we will look a bit closer into connections between operators on E and their restrictions to E_0 . In doing so, the class of operators

$$L_0(E) := \{A \in L(E) : u \in E_0 \Rightarrow Au \in E_0\} \quad (3.7)$$

will turn out to be of particular interest to us.

Lemma 3.13 *For $A \in L(E)$, the condition $A \in L_0(E)$ is equivalent to the strong convergence $Q_n A P_m \rightarrow 0$ as $n \rightarrow \infty$ for every fixed m .*

Proof. Fix an arbitrary $m \in \mathbb{N}$. By Lemma 2.6, the strong convergence $Q_n A P_m \rightarrow 0$ as $n \rightarrow \infty$ is equivalent to $A P_m u \in E_0$ for every $u \in E$. Clearly, $A \in L_0(E)$ implies $A P_m u \in E_0$ for every $u \in E$, since $P_m u \in E_0$. The reverse implication follows from $P_m u \rightarrow u$ for every $u \in E_0$, from the continuity of A on $(E, \|\cdot\|)$, and the closedness of E_0 in this topology. ■

As an immediate consequence of Lemmas 3.7 and 3.13 we get the following.

Corollary 3.14 *Operators in $L(E, \mathcal{P})$ map E_0 into E_0 , i.e. $L(E, \mathcal{P}) \subset L_0(E)$.*

We have just seen that with every operator $A \in L(E, \mathcal{P})$ we can associate the operator $B := A|_{E_0}$ on E_0 , where the latter turns out to be in $L(E_0, \mathcal{P})$ again. The following lemma shows that one can also go the other way:

Lemma 3.15 *Every $B \in S(E_0)$ has a unique extension to an operator $A \in S(E)$, defined by*

$$Au := \lim_{n \rightarrow \infty} B P_n u, \quad u \in E, \quad (3.8)$$

where the limit is understood in the strict topology. It holds that $\|A\| = \|B\|$, and if $B \in SN(E_0)$, $L(E_0, \mathcal{P})$ or $K(E_0, \mathcal{P})$, then $A \in SN(E)$, $L(E, \mathcal{P})$ or $K(E, \mathcal{P})$, respectively. Conversely, if $A \in S(E)$, $SN(E)$, $L(E, \mathcal{P})$ or $K(E, \mathcal{P})$ and if $A(E_0) \subset E_0$, then $B := A|_{E_0} \in S(E_0)$, $SN(E_0)$, $L(E_0, \mathcal{P})$ or $K(E_0, \mathcal{P})$, respectively.

Proof. It is easy to see that every sequentially continuous linear operator on the TVS (E_0, s) has a unique sequentially continuous extension to the sequential completion (E, s) of the TVS. The construction of this extension in our case is

given by (3.8). For $u \in E_0$, we have $BP_nu \xrightarrow{s} Bu$ since $P_nu \xrightarrow{s} u$ and $A \in S(E_0)$, so that $A|_{E_0} = B$. Moreover, for $u \in E$,

$$\|Au\| \leq \sup_n \|BP_nu\| \leq \|B\| \sup_n \|P_nu\| = \|B\| \|u\|$$

so that A is bounded and $\|A\| \leq \|B\|$. Together with $A|_{E_0} = B$, this gives $\|A\| = \|B\|$.

Now let us show that $A \in SN(E)$ if $B \in SN(E_0)$. For every $u \in E$ and $k, n \in \mathbb{N}$, it holds that $Q_nP_ku \in E_0$ since $P_ku \in E_0$ and $Q_n \in L_0(E)$, and hence $BQ_nP_ku = AQ_nP_ku \rightarrow AQ_nu$ as $k \rightarrow \infty$, the latter since $P_ku \xrightarrow{s} u$ and $A, Q_n \in S(E)$. Thus, for $u \in E$ with $\|u\| = 1$,

$$\|AQ_nu\| \leq \sup_k \|BQ_nP_ku\| \leq \sup_k \|BQ_nP_k\| \leq \|BQ_n\| \sup_k \|P_k\| \rightarrow 0$$

as $n \rightarrow \infty$, by Lemma 3.2, since $B \in SN(E_0)$. Hence $A \in SN(E)$, by Lemma 3.2 again.

From the trivial equality $\|Q_nAP_m\| = \|Q_nBP_m\|$, together with $A \in S(E)$, we get that $A \in L(E, \mathcal{P})$ if $B \in L(E_0, \mathcal{P})$, by Corollary 3.4 and Lemma 3.7. Finally, it follows that $A \in K(E, \mathcal{P}) = L(E, \mathcal{P}) \cap SN(E)$ if $B \in K(E_0, \mathcal{P}) = L(E_0, \mathcal{P}) \cap SN(E_0)$, by Lemma 3.9. ■

3.2.3 Compact Operators on (E, s) and Generalisations

A linear operator on a TVS is said to be *compact* if the image of some neighbourhood of zero is relatively compact. A linear operator is often said to be *Montel* if it has the weaker property that it maps bounded sets onto relatively compact sets. These properties coincide when the TVS is a normed space. Much of the familiar theory of compact operators on normed spaces generalises to compact operators on locally convex separated TVS's, for example the theory of Riesz [151]. In particular, a compact operator has a discrete spectrum (as an element of the algebra of continuous operators), whose only accumulation point is zero, and all non-zero points of the spectrum are eigenvalues. By contrast, as we will see below, the spectrum of a Montel operator may be much more complex.

Let $KS(E)$ denote the set of compact operators on (E, s) and $M(E)$ the set of Montel operators on (E, s) .

$$\begin{array}{lll} KS(E): & \text{neighbourhood} & \rightarrow \text{relatively compact} \\ M(E): & \text{bounded} & \rightarrow \text{relatively compact} \end{array}$$

Then it is standard that $KS(E) \subset M(E) \subset L(E)$ and $KS(E) \subset S(E)$. Also $K(E) \subset M(E)$ since bounded sets coincide in the strict and norm topologies and

relatively compact sets in the norm topology are relatively compact in the strict topology. Thus, by Example 3.12 c), it may not hold that $M(E) \subset S(E)$. A Venn diagram illustrating the various subsets of $L(E)$ that we have introduced by now is shown in Figure 3.2 below.

By Lemmas 2.7 and 2.10, an operator A is in $M(E)$ iff the image of every norm-bounded set is relatively sequentially compact in the strict topology. Operators with this property are termed *sequentially compact with respect to (E, s)* in [46]. The following two lemmas are useful characterisations of $M(E)$ in the case when (E, s) is sequentially complete (which is the case when $E = E^p$ with $p \neq 0$, by Lemma 2.9).

Lemma 3.16 *If $E = E^p$ with $p \neq 0$ then $A \in M(E)$ iff $A \in L(E)$ and $P_m A \in K(E)$ for every $m \in \mathbb{N}$.*

Proof. The lemma follows immediately from the equivalence of (a) and (d) in Lemma 2.10. This implies that $A \in M(E)$ iff $A(S)$ is norm-bounded and $P_m A(S)$ is relatively compact in the norm topology, for every m and every norm-bounded set S . ■

Remark 3.17 Operators A on $L^p(\mathbb{R}^N)$ are termed *locally compact* in [37, 106, 138, 143] if both $P_m A$ and $A P_m$ are compact operators for each m , where now P_m is the operator of multiplication by the characteristic function of $[-m, m]^N \subset \mathbb{R}^N$. In the case $BC(\mathbb{R}^N)$ of Example 2.4, an operator A is termed *locally compact* in [86] if it holds merely that $P_m A$ is compact for every m , which corresponds, by Lemma 3.16, to what we call a Montel operator. □

Lemma 3.18 *If $A \in M(E)$ then $A P_n \in KS(E)$ for every $n \in \mathbb{N}$. Conversely, if $p \neq 0$, $A \in S(E)$ and $A P_n \in M(E)$ for every n , then $A \in M(E)$.*

Proof. By Lemma 3.2, $P_n \in SN(E)$, and so, by Lemma 3.5, maps some neighbourhood in (E, s) to a bounded set in $(E, \|\cdot\|)$. (In fact every neighbourhood in (E, s) is mapped to a bounded set.) Thus $A P_n \in KS(E)$ if $A \in M(E)$.

If $A P_n \in M(E)$ for every n then, by Lemma 3.16, $P_m A P_n \in K(E)$ for every m and n . If also $A \in S(E)$ then, by Corollary 3.4, $\|P_m A - P_m A P_n\| \rightarrow 0$ as $n \rightarrow \infty$, so that $P_m A \in K(E)$ for every m . Thus $A \in M(E)$ by Lemma 3.16. ■

That being Montel on (E, s) is significantly weaker than being compact is very clear in the case when P_n is compact for all n . The next two results follow from Corollary 2.11 (the first is also a corollary of Lemma 3.16).

Corollary 3.19 *If $p \neq 0$ and $P_n \in K(E)$ for every n then $M(E) = L(E)$.*

$M(E)$ is the set of operators which map bounded sets to relatively compact sets in (E, s) , and we have seen in Lemma 3.5 that $SN(E)$ is precisely the set of those operators that map some neighbourhood in (E, s) to a bounded set. On the other hand, $KS(E)$ is the set of those operators that map some neighbourhood to a relatively compact set:

Clearly, if $A \in SN(E) \cap M(E)$ then $A^2 \in KS(E)$. What is less clear is that $A \in KS(E)$, which is the content of the next lemma (Lemma 3.27 in [39]).

We finish the section with examples of operators in $S(E)$, $KS(E)$, and $M(E)$.

$$g(x) = (\kappa \star f)(x) = \int_{\mathbb{R}^N} \kappa(x-y) f(y) dy, \quad x \in \mathbb{R}^N.$$

By the identification (2.4) of $L^\infty(\mathbb{R}^N)$ with $E = \ell^\infty(\mathbb{Z}^N, L^\infty([0, 1]^N))$, the above operator $f \mapsto \kappa \star f$ is identified with an operator $K : E \rightarrow E$. It is well-known [133, 86] that its spectrum is $\{0\} \cup \{\hat{\kappa}(\xi) : \xi \in \mathbb{R}^N\}$, where $\hat{\kappa} \in BC(\mathbb{R}^N)$ is the Fourier transform of κ . All non-zero points of the spectrum are eigenvalues (where $\hat{\kappa}(\xi)$ has eigenfunction $u(s) := \exp(i\xi \cdot s)$). Since the spectrum of K is not discrete, $K \notin KS(E)$. But it is not hard to see that $K \in M(E) \cap S(E)$. \square

Example 3.22 (Cf. [6, 40].) Consider the integral operator $f \mapsto g$ on $L^\infty(\mathbb{R})$ with

$$g(x) = \int_0^1 \exp(ixy) f(y) dy, \quad x \in \mathbb{R}$$

and let K be the corresponding operator, via (2.4), on $E = \ell^\infty(\mathbb{Z}, L^\infty([0, 1]))$. Then $K = KP_n$ for all $n \geq 1$, so that $K \in KS(E)$ by Lemma 3.18. But $K \notin K(E)$ as, defining $f_n(x) = \exp(-inx)$, $x \in \mathbb{R}$, the sequence (Kf_n) has no norm-convergent subsequence since $(Kf_n)(x) \rightarrow 0$ as $x \rightarrow \infty$ for every n but $(Kf_n)(n) = 1$ for each n . \square

3.2.4 Algebraic Properties

We will find the algebraic properties collected in the following lemma useful. These are immediate from the definitions and Lemmas 3.5, 3.18 and 3.20.

Lemma 3.23 *Let A and B be linear operators on E . Then*

$$\begin{aligned} A \in M(E), \quad B \in L(E) &\Rightarrow AB \in M(E) \\ A \in S(E), \quad B \in M(E) &\Rightarrow AB \in M(E) \\ A \in L(E), \quad B \in SN(E) &\Rightarrow AB \in SN(E) \\ A \in SN(E), \quad B \in S(E) &\Rightarrow AB \in SN(E) \\ A \in SN(E), \quad B \in M(E) &\Rightarrow AB \in K(E) \\ A \in M(E), \quad B \in SN(E) &\Rightarrow AB \in KS(E) \\ A \in KS(E) &\Rightarrow A^2 \in K(E) \end{aligned}$$

$S(E)$, $SN(E)$, $M(E)$, and $KS(E)$ are all vector subspaces of $L(E)$. It follows from the above lemma that they are all subalgebras of $L(E)$, and that $SN(E)$, $M(E) \cap S(E)$, and $KS(E)$ are all (two-sided) ideals of $S(E)$. Moreover, all these subalgebras are closed when endowed with the norm topology of $L(E)$.

Lemma 3.24 *$S(E)$, $SN(E)$, $M(E)$, and $KS(E)$ are all Banach subalgebras of $L(E)$, with $S(E)$ a unital subalgebra and $SN(E)$, $M(E) \cap S(E)$, and $KS(E)$ ideals (two-sided, closed) of $S(E)$.*

Proof. It only remains to show that each subalgebra is closed. If $A \in L(E)$ is in the closure of $SN(E)$ then, for every $B \in SN(E)$ and every n ,

$$\|AQ_n\| \leq \|BQ_n\| + \|(B - A)Q_n\| \leq \|BQ_n\| + 2\|B - A\|.$$

Since B can be chosen to make $\|B - A\|$ arbitrarily small and, by Lemma 3.2, $\|BQ_n\| \rightarrow 0$ as $n \rightarrow \infty$ for every B , it follows that $\|AQ_n\| \rightarrow 0$ as $n \rightarrow \infty$ so that $A \in SN(E)$. Thus $SN(E)$ is closed.

Since $SN(E)$ is closed it follows from Lemma 3.3 that $S(E)$ is closed.

As $K(E)$ is closed, it follows from Lemma 3.16 in the case $p \neq 0$ that $M(E)$ is closed. In the general case, to see that $M(E)$ is closed, suppose that $(A_m) \subset M(E)$ and $A_m \rightrightarrows A \in L(E)$. Let (u_n) be a bounded sequence in E . Then, by a diagonal argument as in the proof of Lemma 3.27 of [39], we can find a subsequence, denoted again by (u_n) , such that, for each m , there exists a $v_m \in E$ such that $A_m u_n \xrightarrow{s} v_m$ as $n \rightarrow \infty$. Arguing as in the proof of Lemma 3.27 of [39], we can show that the sequence (v_m) is Cauchy in $(E, \|\cdot\|)$ and so has a limit $v \in E$, and that $Au_n \xrightarrow{s} v$. This shows that the image of every bounded set under A is relatively sequentially compact and so relatively compact in (E, s) , by Lemma 2.10, i.e. $A \in M(E)$.

By Lemma 3.20, $KS(E) = M(E) \cap SN(E)$, being the intersection of two closed spaces, is closed itself. ■

We have already mentioned that an operator $A \in L(E)$ is invertible iff it is a bijective map $E \rightarrow E$ and that the subalgebra $L(E, \mathcal{P})$ is inverse closed in $L(E)$. An interesting question is whether also $S(E)$ is inverse closed, i.e. whether, if $A \in S(E)$ is invertible, it necessarily holds that $A^{-1} \in S(E)$. Since (E, s) is not barrelled [46], this question is not settled by standard generalisations of the open mapping theorem to non-metrisable TVS's [151]. Indeed, it is not clear to us whether $S(E)$ is inverse closed without further assumptions on $E = E^p(X)$. But we do have the following result which implies that $S(E)$ is inverse closed in the case when $p \neq 0$ and X is finite-dimensional.

Lemma 3.25 *Suppose $A, B \in S(E)$ are invertible and that $A^{-1} \in S(E)$ and $A - B \in M(E)$. Then $B^{-1} \in S(E)$.*

Proof. We have that $B^{-1} = D^{-1}A^{-1}$, where $D = I + C$ and $C = A^{-1}(B - A)$. By Lemma 3.23, $C \in S(E) \cap M(E)$. To show that $B^{-1} \in S(E)$ we need only to show that $D^{-1} \in S(E)$.

Suppose that $(u_n) \subset E$, $u \in E$, and $u_n \xrightarrow{s} u$. Let $v_n := D^{-1}u_n$. By (2.6), and since $D^{-1} = B^{-1}A \in L(E)$, (u_n) and (v_n) are bounded. For each n ,

$$v_n + Cv_n = u_n. \tag{3.9}$$

Since $C \in M(E)$ there exists a subsequence (v_{n_m}) and $v \in E$ such that $u_{n_m} - Cv_{n_m} \xrightarrow{s} v$. From (3.9) it follows that $v_{n_m} \xrightarrow{s} v$. Since $C \in S(E)$, it follows that $u_{n_m} - Cv_{n_m} \xrightarrow{s} u - Cv$. Thus $v = u - Cv$, i.e. $v = D^{-1}u$. We have shown that $v_n = D^{-1}u_n$ has a subsequence strictly converging to $v = D^{-1}u$. By the same argument, every subsequence of v_n has a subsequence strictly converging to v . Thus $D^{-1}u_n \xrightarrow{s} D^{-1}u$. So $D^{-1} \in S(E)$. ■

Corollary 3.26 *If $E = E^p(X)$ with $p \neq 0$ and $\dim X < \infty$ then $S(E)$ is inverse closed.*

Proof. If $p \neq 0$ and $\dim X < \infty$, and $B \in S(E)$ is invertible, then $I - B \in M(E)$ by Corollary 3.19, so that $B^{-1} \in S(E)$ by the above lemma. ■

3.3 Duality: Adjoint and Preadjoint Operators

3.3.1 Definitions

The *dual space* of a Banach space X , that is the space of all bounded and linear functionals on X , is denoted by X^* . Moreover, if there exists one, then by X^\diamond we denote a Banach space whose dual space is (isometrically isomorphic to) X , and we will refer to X^\diamond as a *predual space* of X .

Remark 3.27 Note that in general, neither existence nor uniqueness (up to isometrical isomorphy, of course) of a predual space is guaranteed. For example, $L^1(\mathbb{R})$ does not possess any predual spaces, whereas ℓ^1 has the two different predual spaces c and c_0 , the spaces of all convergent and all null sequences, respectively. However, it was first pointed out by Grothendieck [78] that all ℓ^∞ and L^∞ spaces have a unique predual, and this observation was generalised to von Neumann algebras later in [159]. □

Remember that X can be identified with a subspace \hat{X} of $X^{**} := (X^*)^*$ by the mapping

$$x \in X \mapsto \hat{x} \in \hat{X} \quad \text{with} \quad \hat{x}(f) = f(x) \quad \forall f \in X^*. \quad (3.10)$$

As usual, we call X *reflexive*, if (3.10) is a bijection between X and X^{**} so that X is isomorphic to X^{**} . Moreover, recall that for $A \in L(X)$, the *adjoint operator* $A^* \in L(X^*)$ is defined by

$$(A^*f)(x) = f(Ax) \quad \text{for all} \quad x \in X \quad \text{and} \quad f \in X^*.$$

It is well-known (e.g. [156]) that, for $p \in [1, \infty)$, the dual space of $E^p(X)$ can be identified with $E^q(X^*)$, where $1/p + 1/q = 1$ and $1/\infty = 0$, and that $(E^0(X))^*$

can be identified with $E^1(X^*)$. Unfortunately, the dual of $E^\infty(X)$ is outside this class of spaces; it is strictly larger¹ than $E^1(X^*)$. That makes the identification and study of the adjoint operator A^* of $A \in L(E^p(X))$ much more difficult for $p = \infty$ than for $p < \infty$. For some arguments in the case $p = \infty$, where the aspect of duality is important, we will therefore need to find an adequate substitute for the adjoint operator A^* .

Fix a Banach space E , and by F denote a predual space of E . In the case we have in mind, $E = E^\infty(X)$ and $F = E^1(X^\triangleleft)$ provided that X^\triangleleft exists (see e.g. [159]). If $A \in L(E)$, $F^* \cong E$ and if there exists an operator $B \in L(F)$ such that

$$B^* = A, \quad (3.11)$$

we will refer to B as a *preadjoint operator* of A , an operator whose adjoint equals A , and we will frequently denote B by A^\triangleleft . In many situations we will restrict ourselves to operators A on E that possess a preadjoint operator.

There is an alternative and equivalent characterization of those operators $A \in L(E)$ that possess a preadjoint operator. Again suppose that a predual space F of E exists and remember that F can be identified with $\hat{F} \subset F^{**} \cong E^*$.

If the adjoint operator A^* , acting on E^* , maps the subspace $\hat{F} \subset E^*$ into \hat{F} again, we can define an operator $B \in L(F)$ by

$$\widehat{Bf} = A^* \hat{f} \quad \forall f \in F. \quad (3.12)$$

Lemma 3.28 *If F is a Banach space, $F^* \cong E$, and $A \in L(E)$, then $B \in L(F)$ is the preadjoint of A , i.e. (3.11) holds, iff $A^*(\hat{F}) \subset \hat{F}$ and (3.12) holds.*

Proof. For arbitrary elements $e \in E$ and $f \in F$ and arbitrary operators $A \in L(E)$ and $B \in L(F)$, one has

$$(Ae)(f) = \hat{f}(Ae) = (A^* \hat{f})(e)$$

and

$$(B^*e)(f) = e(Bf) = (\widehat{Bf})(e).$$

Consequently, (3.11) implies (3.12), and vice versa. ■

Remark 3.29 Another equivalent characterisation, besides $A^*(\hat{F}) \subset \hat{F}$, for the existence of a preadjoint of A is that A is continuous in the weak-* topology on E (see e.g. [159]). □

¹See Example 3.12 c) for a functional on $E^\infty(X)$ which does not correspond to an element of $E^1(X^*)$.

We denote the set of all $A \in L(E)$ that possess a preadjoint operator $A^\natural \in L(F)$ by

$$L^\natural(E) := \left\{ A = B^* \in L(E) : B \in L(F) \right\} = \left\{ A \in L(E) : A^*(\hat{F}) \subset \hat{F} \right\}.$$

So for $A \in L^\natural(E)$, we can pass from $L(E)$ to $L(F)$ by $A \mapsto A^\natural$ and back to $L(E)$ by $B \mapsto B^*$. From basic properties of the adjoint operator it follows that A^\natural is invertible in $L(F)$ if and only if $(A^\natural)^* = A$ is invertible in $L(E)$. Moreover,

$$\|A^\natural\|_{L(F)} = \|(A^\natural)^*\|_{L(F^*)} = \|A\|_{L(E)}. \quad (3.13)$$

Proposition 3.30 $L^\natural(E)$ is an inverse closed Banach subalgebra of $L(E)$.

Proof. If $A_1 = B_1^*$ and $A_2 = B_2^*$ are in $L^\natural(E)$, then also $A_1 + A_2 = (B_1 + B_2)^*$ and $A_1 A_2 = (B_2 B_1)^*$ are in $L^\natural(E)$. If $(A_k) \subset L^\natural(E)$ tends to A in the norm of $L(E)$, then, by (3.13), not only (A_k) is a Cauchy sequence in $L(E)$ but also the sequence $(B_k) = (A_k^\natural)$ is a Cauchy sequence in $L(F)$. Let B denote the norm limit of B_k . Then, by (3.13) again, $A_k = B_k^* \rightrightarrows B^*$. Together with $A_k \rightrightarrows A$, this shows that $A = B^* \in L^\natural(E)$.

To see that $L^\natural(E)$ is inverse closed, take an arbitrary invertible operator $A \in L^\natural(E)$. But then also $B = A^\natural$ is invertible, and $A^{-1} = (B^*)^{-1} = (B^{-1})^* \in L^\natural(E)$. ■

3.3.2 Duality in Action: Fredholm Operators on E^∞

We will stick to the case $E := E^\infty(X)$ for this section, where X is an arbitrary complex Banach space. We now exploit duality a bit more to establish connections between Fredholm properties of an operator A on E and its restriction $A_0 := A|_{E_0}$ on $E_0 = E^0(X)$. As an auxiliary space we will also look at $E_1 := E^1(X^*) \cong (E_0)^*$. If X is reflexive, i.e. $X \cong X^{**}$, one has that

$$(E_0)^{**} \cong (E_1)^* \cong E^\infty(X^{**}) \cong E^\infty(X) = E.$$

In the general case when X is not reflexive we still can, in a natural way, isometrically embed E as a closed subspace of $(E_0)^{**}$. On the other hand, we can always embed $(E_0)^* \cong E_1 = E^1(X^*)$ into the dual space of $E = E^\infty(X)$. In terms of these two embeddings, we show that every $A \in S(E)$ is the restriction of A_0^{**} to E and that A_0^* is the restriction of A^* to E_1 , yielding close connections between Fredholmness of A on E and A_0 on E_0 . Further connections are established in the case that X has a predual X^\natural and A has a preadjoint A^\natural on $E^\natural = E^1(X^\natural)$.

For $u = (u_j)_{j \in \mathbb{Z}^N} \in E = E^\infty(X)$ and $v = (v_j)_{j \in \mathbb{Z}^N} \in E_1 = E^1(X^*)$, define the bilinear form (\cdot, \cdot) on (E, E_1) by

$$(u, v) := \sum_{j \in \mathbb{Z}^N} v_j(u_j), \quad (3.14)$$

and note that, equipped with (\cdot, \cdot) , the pair (E, E_1) is a dual system in the sense e.g. of [86]. If $u \in E_0$ and $v \in E_1 = E_0^*$ then $(u, v) = v(u)$. A similar equation holds if X has a predual space X^\diamond . Then E is the dual space of $E^\diamond := E^1(X^\diamond)$. Denote by J_X the canonical embedding of X^\diamond into its second dual X^* , given by $(J_X y)(x) = x(y)$, $y \in X^\diamond$, $x \in X$, and let $J^\diamond : E^1(X^\diamond) \rightarrow E^1(X^*)$ be the natural embedding $J^\diamond u = (J_X u_j)_{j \in \mathbb{Z}^N}$. Note that both J_X and J^\diamond are isometries. Then

$$(u, J^\diamond v) = u(v), \quad u \in E, v \in E^\diamond. \quad (3.15)$$

A simple but important observation is that, if $(u_n) \subset E$, $u \in E$, and $v \in E_1$, then

$$u_n \xrightarrow{s} u \quad \Rightarrow \quad (u_n, v) \rightarrow (u, v). \quad (3.16)$$

For every $A \in L_0(E)$ (recall the definition (3.7) of $L_0(E)$) let $A_0 \in L(E_0)$ be defined by $A_0 := A|_{E_0}$. Then its adjoint $A_0^* \in L(E_0^*) = L(E_1)$. From Corollary 3.14 we recall that, in particular, $A \in L_0(E)$ if $A \in L(E, \mathcal{P}) \subset S(E)$.

Lemma 3.31 *If $A \in S(E) \cap L_0(E)$ then*

$$(Au, v) = (u, A_0^* v), \quad u \in E, v \in E_1,$$

i.e. A is the transpose of A_0^ with respect to the dual system (E, E_1) .*

Proof. For $u \in E_0$, $v \in E_0^* = E_1$,

$$(Au, v) = (A_0 u, v) = (u, A_0^* v).$$

Thus, for $u \in E$, $v \in E_1$,

$$(AP_n u, v) = (P_n u, A_0^* v).$$

Taking the limit as $n \rightarrow \infty$, in view of $P_n u \xrightarrow{s} u$, $A \in S(E)$ and (3.16), the result follows. ■

Now define the promised embedding $J : E \rightarrow E_0^{**}$ by

$$(Ju)(v) = (u, v), \quad u \in E, v \in E_0^* = E_1.$$

It is easy to check that J is an isometry, so that E is isometrically isomorphic to $\check{E} := J(E) \subset E_0^{**}$. For $A \in L(E)$ define $\check{A} \in L(\check{E})$ by $\check{A} := JAJ^{-1}$.

Lemma 3.32 *If $A \in S(E) \cap L_0(E)$, then $A_0^{**}(\check{E}) \subset \check{E}$ and $\check{A} = A_0^{**}|_{\check{E}}$, so that*

$$\alpha(A_0) \leq \alpha(A) = \alpha(\check{A}) \leq \alpha(A_0^{**}).$$

Proof. For $u \in \check{E}$, $v \in E_1$, with $z := J^{-1}u \in E$,

$$\check{A}u(v) = (J(Az))(v) = (Az, v) = (z, A_0^*v),$$

by Lemma 3.31, and

$$A_0^{**}u(v) = u(A_0^*v) = (Jz)(A_0^*v) = (z, A_0^*v).$$

■

To make full use of the above observation, we need the following characterisation, for a Banach space Z , of those operators $C \in L(Z)$ whose range is closed, which is a standard corollary of the open mapping theorem (applied to the injective operator $z + \ker C \mapsto Cz$ from $Z/\ker C$ to Z , also see [72, Theorem XI.2.1]): that

$$C(Z) \text{ is closed} \quad \Leftrightarrow \quad \exists c > 0 \text{ s.t. } \|Cz\| \geq c \inf_{v \in \ker C} \|z - v\|, \quad \forall z \in Z. \quad (3.17)$$

We also need the following consequence of the above characterisation.

Lemma 3.33 *Suppose that Z is a Banach space, Z_0 is a closed subspace of Z , $C(Z_0) \subset Z_0$, and set $C_0 := C|_{Z_0}$. If the range of C is closed and $\ker C = \ker C_0$ (i.e. $\ker C \subset Z_0$), then the range of C_0 is also closed.*

Proof. If the conditions of the lemma are satisfied then, by (3.17), there exists $c > 0$ such that $\|Cz\| \geq c \inf_{v \in \ker C} \|z - v\|$, $z \in Z$. But, since $Z_0 \subset Z$ and $\ker C = \ker C_0$, this implies that $\|C_0z\| \geq c \inf_{v \in \ker C_0} \|z - v\|$, $z \in Z_0$, so that the range of C_0 is closed. ■

Corollary 3.34 *If $A \in S(E) \cap L_0(E)$ and A_0 is semi-Fredholm with $\alpha(A_0) < \infty$, then A is semi-Fredholm and $\ker A = \ker A_0$.*

Proof. If the conditions of the lemma are satisfied then, from standard results on Fredholm operators (e.g. [86]), we have that A_0^* and A_0^{**} are also semi-Fredholm, and $\alpha(A_0) = \beta(A_0^*) = \alpha(A_0^{**})$. Applying Lemma 3.32, it follows that $\alpha(A_0) = \alpha(A) = \alpha(A_0^{**})$. Further, since $\alpha(A_0)$ is finite and $\ker A_0 \subset \ker A$, $\ker \check{A} \subset \ker A_0^{**}$, it follows that $\ker A = \ker A_0$ and that $\ker \check{A} = \ker A_0^{**}$. Applying Lemma 3.33, since the range of A_0^{**} is closed it follows that the range of \check{A} is closed and so $A(E)$ is closed, and A is semi-Fredholm. ■

We will prove the converse result in the case when X has a predual space X^\diamond and A has a preadjoint A^\diamond on $E^\diamond = E^1(X^\diamond)$.

Recalling the isometry $J^\diamond : E^\diamond \rightarrow E_1$ introduced above, let $\check{E}^\diamond = J^\diamond(E^\diamond) \subset E_1 = E_0^*$, so that \check{E}^\diamond is isometrically isomorphic to E^\diamond . For $A^\diamond \in L(E^\diamond)$ let $\check{A}^\diamond \in L(\check{E}^\diamond)$ be defined by $\check{A}^\diamond = J^\diamond A^\diamond (J^\diamond)^{-1}$.

Lemma 3.35 *If $A \in L_0(E)$, X has a predual X^\diamond and A a preadjoint $A^\diamond \in L(E^\diamond)$, then $A_0^*(\check{E}^\diamond) \subset \check{E}^\diamond$ and $\check{A}^\diamond = A_0^*|_{\check{E}^\diamond}$, so that $\ker \check{A}^\diamond \subset \ker A_0^*$.*

Proof. For $u \in \check{E}^\diamond$ and $v \in E_0$, where $z := (J^\diamond)^{-1}u \in E^\diamond$, using (3.15),

$$\check{A}^\diamond u(v) = (v, \check{A}^\diamond u) = v(A^\diamond z) = Av(z) = (Av, u).$$

Also,

$$A_0^* u(v) = u(A_0 v) = (A_0 v, u) = (Av, u).$$

■

Let $J_1 : E_1 \rightarrow E^*$ be defined by

$$J_1 u(v) := Jv(u) = (v, u), \quad u \in E_1, v \in E.$$

It is easy to check that J_1 is also an isometry. Let $\check{E}_1 := J_1(E_1) \subset E^*$, which is isometrically isomorphic to E_1 . For $A_1 \in L(E_1)$ let $\check{A}_1 \in L(\check{E}_1)$ be defined by $\check{A}_1 := J_1 A_1 J_1^{-1}$.

Lemma 3.36 *If $A \in S(E) \cap L_0(E)$, then $A^*(\check{E}_1) \subset \check{E}_1$ and $\check{A}_0^* = A^*|_{\check{E}_1}$, so that $\ker \check{A}_0^* \subset \ker A^*$.*

Proof. For $u \in \check{E}_1$, $v \in E$, where $w = J_1^{-1}u \in E_1$,

$$\check{A}_0^* u(v) = J_1(A_0^* w)(v) = (v, A_0^* w) = (Av, w),$$

by Lemma 3.31. Also,

$$A^* u(v) = u(Av) = J_1 w(Av) = (Av, w).$$

■

We note that if the conditions of Lemmas 3.35 and 3.36 are satisfied, then

$$\alpha(A^\diamond) \leq \alpha(A_0^*) \leq \alpha(A^*). \quad (3.18)$$

Proposition 3.37 *Suppose that $A \in S(E) \cap L_0(E)$, X has a predual X^\natural and A has a preadjoint $A^\natural \in L(E^\natural)$. Then A is Fredholm if and only if A_0 is Fredholm and, if they are both Fredholm, then $\alpha(A_0) = \alpha(A)$, $\beta(A_0) = \beta(A)$, and $\ker A = \ker A_0$.*

Proof. Suppose first that A_0 is Fredholm. Then, by Corollary 3.34, A is semi-Fredholm and $\ker A = \ker A_0$. This implies that A^\natural and A^* are also semi-Fredholm, and so, and using (3.18),

$$\beta(A) = \alpha(A^\natural) \leq \alpha(A_0^*) = \beta(A_0),$$

so that A is Fredholm. Moreover,

$$\beta(A) = \alpha(A^*) \geq \alpha(A_0^*) = \beta(A_0)$$

so $\beta(A) = \beta(A_0)$.

Conversely, if A is Fredholm then so are A^\natural and A^* and $\alpha(A^\natural) = \beta(A) = \alpha(A^*)$. Thus, by (3.18), $\alpha(\check{A}_0^*) = \alpha(A_0^*) = \alpha(A^*)$ is finite and so it follows from Lemma 3.36 that $\ker \check{A}_0^* = \ker A^*$. Applying Lemma 3.33 we see that the range of \check{A}_0^* is closed, so that the range of A_0^* is closed and A_0^* is semi-Fredholm. Thus A_0 is also semi-Fredholm, with $\beta(A_0) = \alpha(A_0^*) = \alpha(A^*) < \infty$. But also $\alpha(A_0) \leq \alpha(A)$ is finite, so A_0 is Fredholm. ■

Note that the above proposition and its proof simplifies greatly if the Banach space X is reflexive, in particular if X is finite dimensional. For then we can choose $X^\natural = X^*$ so that $E^\natural = E_1$ and $E_0^{**} = E$. Note also that, if the conditions of the above proposition hold, in particular if A has a preadjoint, then the above proposition implies that A is invertible if and only if A_0 is invertible. But even without existence of a preadjoint, we can prove this result in some cases; an observation which will be useful to us later.

Lemma 3.38 *If $A \in S(E) \cap L_0(E)$ and A is invertible, then A_0 is invertible.*

Proof. If A is invertible then A_0 is injective and it follows from Lemma 3.33 that the range of A_0 is closed. Further, since A is the transpose of A_0^* with respect to the dual system (E, E_1) it follows (see e.g. [86]) that $0 = \beta(A) \geq \alpha(A_0^*) = \beta(A_0)$. Thus A_0 is surjective. ■

Lemma 3.39 *Suppose that $A \in L(E, \mathcal{P})$ or that $A = I + K$ with $K \in S(E) \cap M(E) \cap L_0(E)$, and suppose that A_0 is invertible. Then A is invertible.*

Proof. If the conditions of the lemma apply then, by Corollary 3.34, A is injective. In the case that $A \in L(E, \mathcal{P})$ then $A_0 \in L(E_0, \mathcal{P})$ by Lemma 3.15, and since $L(E_0, \mathcal{P})$ is inverse closed (Theorem 1.1.9 of [143]), we have that $A_0^{-1} \in S(E_0)$. This holds also by a modification of the proof of Lemma 3.25 in the case that $A = I + K$ with $K \in S(E) \cap M(E) \cap L_0(E)$. For if $(u_n) \subset E_0$, $u \in E_0$, and $u_n \xrightarrow{s} u$ then, defining $v_n := A_0^{-1}u_n$,

$$v_n + Kv_n = u_n \quad (3.19)$$

holds, and since $K \in M(E)$ there exists a subsequence (v_{n_m}) and $v \in E$ such that $u_{n_m} - Kv_{n_m} \xrightarrow{s} v$. From (3.19) it follows that $v_{n_m} \xrightarrow{s} v$. Since $K \in S(E)$, it follows that $u_{n_m} - Kv_{n_m} \xrightarrow{s} u - Kv$. Thus $v = u - Kv$, i.e. $Av = u$. Note that, by injectivity of A , there is only one $v \in E$ with $Av = u$ and that is $v = A_0^{-1}u \in E_0$. We have shown that $v_n = A_0^{-1}u_n$ has a subsequence strictly converging to $v = A_0^{-1}u$. By the same argument, every subsequence of v_n has a subsequence strictly converging to v . Thus $A_0^{-1}u_n \xrightarrow{s} A_0^{-1}u$. So $A_0^{-1} \in S(E_0)$.

Let $B \in S(E)$ be the unique extension of A_0^{-1} from E_0 to E , which exists by Lemma 3.15. Then, for every $u \in E$,

$$BAu = s\text{-}\lim_{n \rightarrow \infty} BAP_nu = s\text{-}\lim A_0^{-1}A_0P_nu = u$$

and, similarly, $ABu = u$. So A is invertible. ■

Corollary 3.40 *For $A \in L(E, \mathcal{P})$ it holds that A is invertible iff A_0 is invertible. When both are invertible, then $(A_0)^{-1} = (A^{-1})|_{E_0}$.*

Proof. The first sentence follows immediately from the previous two lemmas, and the equality concerning the two inverses is obvious if both A and A_0 are invertible. ■

3.4 Operator Chemistry

The following beautiful rhetoric picture is drawn in [81] and it reflects nicely an important part of our philosophy: In chemistry, one splits molecules into their elementary parts – the atoms – to understand their properties and to create new molecules, for example. Similarly, it is often useful to think of a ‘complicated’ operator (an ‘operator molecule’) as being composed, via addition, multiplication, taking limits or other operations, of more elementary operators (the ‘operator atoms’).

Let $E = E^p(X)$ with $p \in \{0\} \cup [1, \infty]$ and X some complex Banach space. To introduce our chemistry lab for this text, it is enough to introduce the atoms and the rules after which they can be assembled. In terms of atoms, all we need are the following two types:

Definition 3.41 *If $b = (b(m))_{m \in \mathbb{Z}^N}$ is a bounded sequence of operators $b(m) \in L(X)$ then by M_b we denote the generalised multiplication operator, acting on every $u \in E$ by*

$$(M_b u)(m) = b(m)u(m) \quad \text{for all } m \in \mathbb{Z}^N.$$

Often we will refer to b as the symbol of M_b .

Definition 3.42 *For every $k \in \mathbb{Z}^N$, we denote the shift operator on E by V_k , acting by*

$$(V_k u)(m) = u(m - k) \quad \text{for all } k \in \mathbb{Z}^N,$$

i.e. shifting the whole sequence $u \in E$ by the vector $k \in \mathbb{Z}^N$.

The assembly of these operators will be done in terms of addition, multiplication (meaning composition) and taking the limit in the operator norm. We start with the first two operations: The set of all finite sum-products of generalised multiplication operators and shift operators is a subalgebra of $L(E)$ that will be denoted by $BO(E)$. If we pass to the closure of $BO(E)$ in $L(E)$, i.e. we also pass to norm limits, then we get a Banach subalgebra of $L(E)$ that shall be denoted by $BDO(E)$. We call $BO(E)$, resp. $BDO(E)$, the algebra, resp. Banach algebra, generated by generalised multiplication and shift operators.

Lemma 3.43 *If $E = E^p(X)$ with X reflexive then $BDO(E) \subset L^\natural(E)$.*

Proof. If X is reflexive then $X^\natural \cong X^*$ so that every $B \in L(X)$ has a preadjoint $B^\natural \in L(X^\natural)$ that, by Lemma 3.28, can be identified with B^* acting on $X^\natural \cong X^*$. As a consequence, every generalised multiplication operator M_b with $b = (b(m))_{m \in \mathbb{Z}^N} \in \ell^\infty(\mathbb{Z}^N, L(X))$ has a preadjoint, namely M_{b^*} with $b^* = (b(m)^*)_{m \in \mathbb{Z}^N} \in \ell^\infty(\mathbb{Z}^N, L(X^*))$. By $V_k = V_{-k}^*$ it is clear that also every shift operator has a preadjoint. From Proposition 3.30 and the definition of $BDO(E)$ we get that every $A \in BDO(E)$ has a preadjoint if X is reflexive. ■

The elements of $BO(E)$ and $BDO(E)$ are called *band operators* and *band-dominated operators* on E , respectively. Note that the class $BO(E)$ does not

depend on the value of the parameter p in $E = E^p(X)$, whereas $BDO(E)$ heavily does.

From the simple equalities $M_b M_c = M_{bc}$, $V_k M_b = M_{V_k b}$ and $V_j V_k = V_{j+k}$ for all $b, c \in E^\infty(L(X))$ and $j, k \in \mathbb{Z}^N$ it follows that every element of $BO(E)$ can be uniquely written as

$$A = \sum_{k \in \mathbb{Z}^N, |k| \leq w} M_{b_k} V_k \quad (3.20)$$

with $w \in \mathbb{N}_0$ and $b_k \in E^\infty(L(X))$ for all k under consideration. The smallest number w in (3.20), that is $\max\{0, |k| : k \in \mathbb{Z}^N, b_k \neq 0\}$, is called the *bandwidth* of A .

For the operator A from (3.20), one clearly has

$$\|A\|_{L(E)} = \left\| \sum_{|k| \leq w} M_{b_k} V_k \right\|_{L(E)} \leq \sum_{k \in \mathbb{Z}^N} \|b_k\|_{E^\infty} =: \|A\|_{\mathcal{W}} \quad (3.21)$$

for all spaces $E = E^p$, where we put $b_k = 0$ if $|k|$ exceeds the bandwidth w of A . With this definition, $\|\cdot\|_{\mathcal{W}}$ turns out to be a norm on $BO(E)$, and by $\mathcal{W} = \mathcal{W}(E)$ we denote the completion of $BO(E)$ with respect to $\|\cdot\|_{\mathcal{W}}$. Equipped with $\|\cdot\|_{\mathcal{W}}$, the set $\mathcal{W}(E)$ is a Banach space. We will come back to $\mathcal{W}(E)$ in the next section, where we will see that it is even a Banach algebra.

Similarly to $BO(E)$, also $\mathcal{W}(E)$ does not depend on the parameter p in $E = E^p$. As a consequence of the definition of $\mathcal{W}(E)$ and inequality (3.21), we get that

$$\|A\|_{L(E)} \leq \|A\|_{\mathcal{W}} \quad (3.22)$$

for all $A \in \mathcal{W}(E)$ and all spaces $E = E^p$. We hence have that $\mathcal{W}(E)$, being the completion of $BO(E)$ in the stronger norm $\|\cdot\|_{\mathcal{W}}$, is contained in $BDO(E)$, the completion of $BO(E)$ in the usual norm of $L(E)$, for all spaces $E = E^p$.

3.5 Notions of Operator Convergence

A component in the arguments to be developed is that one needs some notion of the convergence of a sequence of operators. Recall that, for $(A_n) \subset L(E)$, $A \in L(E)$, we write $A_n \rightrightarrows A$ if $\|A_n - A\| \rightarrow 0$ (as $n \rightarrow \infty$) and $A_n \rightarrow A$ if (A_n) converges strongly to A , in the strong operator topology induced by the norm topology on E , i.e. if $A_n u \rightarrow Au$ for all $u \in E$. Following [106, 143] we introduce also the following definition.

Definition 3.44 We say that a sequence $(A_n) \subset L(E)$ \mathcal{P} -converges to $A \in L(E)$ if, for all $K \in K(E, \mathcal{P})$, both

$$\|(A_n - A)K\| \rightarrow 0 \quad \text{and} \quad \|K(A_n - A)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.23)$$

In this case we write $A_n \xrightarrow{\mathcal{P}} A$ or $A = \mathcal{P}\text{-}\lim A_n$.

The following lemma shows that every \mathcal{P} -convergent sequence is bounded in $L(E)$ and that, conversely, for a bounded sequence (A_n) one has to check property (3.23) only for $K \in \mathcal{P}$ in order to guarantee $A_n \xrightarrow{\mathcal{P}} A$.

Lemma 3.45 Suppose $(A_n) \subset L(E)$ and $A \in L(E)$. Then $A_n \xrightarrow{\mathcal{P}} A$ iff (A_n) is bounded in $L(E)$ and, for all m ,

$$\|(A_n - A)P_m\| \rightarrow 0 \quad \text{and} \quad \|P_m(A_n - A)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.24)$$

Remark 3.46 If we replace $L(E)$ by $L(E, \mathcal{P})$ in this proposition then we get a well-known fact, which is already proven in [143]. The new fact here is that (3.23) implies the boundedness of (A_n) also for arbitrary operators in $L(E)$.

Note that, for A and A_n in $L(E)$, already the second property in (3.23) is sufficient for the boundedness of the sequence (A_n) . The first property in (3.23) does not imply this boundedness, as we can see if we put $A_n = nC$ where C is the operator from Example 3.12 c). \square

Proof. Suppose (A_n) is bounded and (3.24) holds. Then, for all $m \in \mathbb{N}$ and all $K \in K(E, \mathcal{P})$, one has

$$\|K(A_n - A)\| \leq \|K\| \|P_m(A_n - A)\| + \|KQ_m\| \|A_n - A\|,$$

where the first term tends to zero as $n \rightarrow \infty$, and the second one is as small as desired if m is large enough. The first property of (3.23) is shown absolutely analogously.

Conversely, if (3.23) holds for all $K \in K(E, \mathcal{P})$, then (3.24) holds for all $m \in \mathbb{N}$ since $\mathcal{P} \subset K(E, \mathcal{P})$. It remains to show that (A_n) is bounded.

Suppose the converse is true. Without loss of generality, we can suppose that $A = 0$. Now we will successively define two sequences: $(m_i)_{i=1}^\infty \subset \mathbb{N}$ and $(n_k)_{k=0}^\infty \subset \mathbb{N}_0$. We start with $m_1 := 1$ and $n_0 := 0$.

For every $k \in \mathbb{N}$, choose $n_k \in \mathbb{N}$ such that

$$n_k > n_{k-1}, \quad \|A_{n_k}\| > k^2 + 3 \quad \text{and} \quad \|P_{m_k} A_{n_k}\| < 1,$$

the latter possible since $P_{m_k} \in K(E, \mathcal{P})$ and $A_n \xrightarrow{\mathcal{P}} 0$. Then

$$\|Q_{m_k} A_{n_k}\| \geq \|A_{n_k}\| - \|P_{m_k} A_{n_k}\| > k^2 + 3 - 1 = k^2 + 2.$$

Take $u_k \in E$ with $\|u_k\| = 1$ and $\|Q_{m_k} A_{n_k} u_k\| > k^2 + 1$, and choose $m_{k+1} > m_k$ such that $\|P_{m_{k+1}} Q_{m_k} A_{n_k} u_k\| > k^2$, which is possible by (i) $_{\mathcal{P}}$. Consequently,

$$\|P_{m_{k+1}} Q_{m_k} A_{n_k}\| > k^2 \quad \text{for all } k \in \mathbb{N}.$$

Now put

$$K := \sum_{j=1}^{\infty} \frac{1}{j^2} P_{m_{j+1}} Q_{m_j}.$$

Then it is easily seen that $K \in K(E, \mathcal{P})$. But on the other hand, from

$$P_{m_{k+1}} Q_{m_k} P_{m_{j+1}} Q_{m_j} = \begin{cases} P_{m_{k+1}} Q_{m_k}, & j = k, \\ 0, & j \neq k, \end{cases}$$

we get that

$$\|K A_{n_k}\| \geq \|P_{m_{k+1}} Q_{m_k} K A_{n_k}\| = \left\| \frac{1}{k^2} P_{m_{k+1}} Q_{m_k} A_{n_k} \right\| > \frac{k^2}{k^2} = 1$$

for every $k \in \mathbb{N}$, which contradicts $\|K A_n\| \rightarrow 0$ as $n \rightarrow \infty$. ■

Example 3.47 Recall the multiplication operators from Definition 3.41 with $b \in \ell^\infty(\mathbb{Z}^N, L(X))$. Clearly, $\|M_b\| = \|b\|$ holds. It is a straightforward consequence of this equation and Lemma 3.45 that, for a sequence $b_n \in \ell^\infty(\mathbb{Z}^N, L(X))$,

$$\begin{aligned} M_{b_n} \xrightarrow{\mathcal{P}} 0 &\Leftrightarrow \sup_n \|b_n\| < \infty \text{ and } b_n \xrightarrow{s} 0 \\ &\Leftrightarrow \sup_n \|b_n\| < \infty \text{ and } \|b_n(m)\| \rightarrow 0, \forall m \in \mathbb{Z}^N. \end{aligned} \quad (3.25)$$

□

We have seen already that $S(E)$ and $L(E, \mathcal{P})$ are Banach subalgebras of $L(E)$. Both are also closed with respect to \mathcal{P} -convergence.

Lemma 3.48 $S(E)$ and $L(E, \mathcal{P})$ are sequentially closed with respect to \mathcal{P} -convergence.

Proof. First suppose $(A_n) \subset S(E)$, $A \in L(E)$ and $A_n \xrightarrow{\mathcal{P}} A$. Then, if $u_n \xrightarrow{s} 0$, for every k and m , we have

$$\|P_k A u_n\| \leq \|P_k(A - A_m)\| \sup_n \|u_n\| + \|P_k A_m u_n\|.$$

But $\|P_k A_m u_n\| \rightarrow 0$ as $n \rightarrow \infty$ since $A_m \in S(E)$, and $\|P_k(A - A_m)\|$ can be made as small as desired by choosing m large. So we get $Au_n \xrightarrow{s} 0$, and therefore $A \in S(E)$.

Similarly (see Proposition 1.1.17(a) in [143] for the details) we show that also $L(E, \mathcal{P})$ is sequentially closed. ■

The \mathcal{P} -limit of sequences in $L(E, \mathcal{P})$ enjoys the following properties (see e.g. [106, Proposition 1.70]):

Proposition 3.49 *For sequences $(A_n), (B_n) \subset L(E, \mathcal{P})$ with \mathcal{P} -limits A and B , respectively, we have*

$$\begin{aligned} \text{a)} \quad & \|A\| \leq \liminf \|A_n\| \leq \sup \|A_n\| < \infty, \\ \text{b)} \quad & \mathcal{P}\text{-}\lim(A_n + B_n) = A + B, \\ \text{c)} \quad & \mathcal{P}\text{-}\lim(A_n B_n) = AB. \end{aligned}$$

To make use of results from [46] we introduce also the notions of operator convergence used there. For $(A_n) \subset L(E)$ and $A \in L(E)$, let us write $A_n \xrightarrow{s} A$ if, for all $(u_n) \subset E$,

$$u_n \xrightarrow{s} u \implies A_n u_n \xrightarrow{s} Au. \quad (3.26)$$

Call $\mathcal{A} \subset L(E)$ *s-sequentially compact* if, for every sequence $(A_n) \subset \mathcal{A}$, there exists a subsequence (A_{n_m}) and $A \in \mathcal{A}$ such that $A_{n_m} \xrightarrow{s} A$. Note that $A \xrightarrow{s} A$ holds iff $A \in S(E)$. It follows that, if $\mathcal{A} \subset L(E)$ is *s-sequentially compact*, then $\mathcal{A} \subset S(E)$.

A more familiar and related notion of operator convergence is that of strong (or pointwise) convergence. For $(A_n) \subset L(E)$, $A \in L(E)$, we will say that (A_n) *converges to A in the strong operator topology on (E, s)* or simply that it *S-converges*, and write $A_n \xrightarrow{S} A$, if

$$A_n u \xrightarrow{s} Au, \quad u \in E. \quad (3.27)$$

Clearly, the *S*-limit is unique, that is $A_n \xrightarrow{S} A$ and $A_n \xrightarrow{S} B$ implies $A = B$. Hence also the *s*-limit and \mathcal{P} -limit are unique, by Lemma 3.50 and Corollary 3.53 below.

Clearly,

$$A_n \rightarrow A \implies A_n \xrightarrow{S} A. \quad (3.28)$$

The following lemmas explore further properties of and relationships between the notions of operator convergence we have introduced. We will exhibit this relationship through Example 3.51.

Lemma 3.50 Suppose $(A_n) \subset L(E)$, $A \in L(E)$. Then

$$A_n \xrightarrow{s} A \implies A_n \xrightarrow{S} A \text{ and } A \in S(E). \quad (3.29)$$

Further, $A_n \xrightarrow{S} A$ as $n \rightarrow \infty$ iff (A_n) is bounded and $P_m(A_n - A) \rightarrow 0$ as $n \rightarrow \infty$ for all $m \in \mathbb{N}$.

Proof. It is clear from the definitions that $A_n \xrightarrow{s} A$ implies $A_n \xrightarrow{S} A$. That $A_n \xrightarrow{S} A$ implies $A \in S(E)$ is shown in [46, Lemma 3.1]. That $A_n \xrightarrow{S} A$ implies $P_m(A_n - A) \rightarrow 0$ is clear from (2.6), and that it also implies that (A_n) is bounded is shown in [46, Lemma 3.3]. Conversely, if (A_n) is bounded and $P_m(A_n - A) \rightarrow 0$ for each m , then, for every $u \in E$, $(A_n u)$ is bounded and $P_m(A_n u - Au) \rightarrow 0$ for each m , so that $A_n u \xrightarrow{s} Au$ by (2.6). ■

Example 3.51 Let $E = E^p(X)$ and recall the multiplication operator M_b as in Example 3.47. Suppose that $(b_n) \subset \ell^\infty(\mathbb{Z}^N, L(X))$. Then, extending the results of Example 3.47, we see that

$$\begin{aligned} M_{b_n} \rightrightarrows 0 &\Leftrightarrow \|b_n\| = \sup_{m \in \mathbb{Z}^N} \|b_n(m)\| \rightarrow 0, \\ M_{b_n} \xrightarrow{\mathcal{P}} 0 &\Leftrightarrow \sup_n \|b_n\| < \infty \text{ and } \|b_n(m)\| \rightarrow 0, \quad \forall m \in \mathbb{Z}^N, \\ M_{b_n} \xrightarrow{s} 0 &\Leftrightarrow M_{b_n} \xrightarrow{S} 0 \\ &\Leftrightarrow \sup_n \|b_n\| < \infty \text{ and } \|b_n(m)u(m)\| \rightarrow 0, \quad \forall m \in \mathbb{Z}^N, u \in E. \end{aligned}$$

Thus $M_{b_n} \xrightarrow{\mathcal{P}} 0$ requires that each component of b_n converges to zero in norm, while $M_{b_n} \xrightarrow{s} 0$ requires that each component of b_n converges strongly to zero. We have (cf. Corollary 3.58 below) that

$$M_{b_n} \rightrightarrows 0 \Rightarrow M_{b_n} \xrightarrow{\mathcal{P}} 0 \Rightarrow M_{b_n} \xrightarrow{s} 0 \Leftrightarrow M_{b_n} \xrightarrow{S} 0 \Leftarrow M_{b_n} \rightarrow 0.$$

If X is finite-dimensional, then $\xrightarrow{\mathcal{P}}$, \xrightarrow{S} and \xrightarrow{s} all coincide. If $p = \infty$, then \rightarrow is equivalent to \rightrightarrows . If $1 < p < \infty$ and X is finite-dimensional, then \rightarrow coincides with $\xrightarrow{\mathcal{P}}$, \xrightarrow{S} and \xrightarrow{s} . □

Lemma 3.52 Suppose $(A_n) \subset L(E)$ is bounded, $A \in S(E)$, and

$$\|P_m(A_n - A)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

for each m . Then $A_n \xrightarrow{s} A$.

Proof. If the conditions of the lemma hold and $u_n \xrightarrow{s} u$ then $Au_n \xrightarrow{s} Au$ and, by (2.6), $\sup_n \|u_n\| < \infty$, so that $(A_n u_n)$ is bounded, and, for each m ,

$$\|P_m(A_n u_n - Au)\| \leq \|P_m(A_n - A)u_n\| + \|P_m A(u_n - u)\| \rightarrow 0$$

as $n \rightarrow \infty$. Thus, by (2.6), $A_n u_n \xrightarrow{s} Au$. ■

As a corollary of Lemmas 3.45 and 3.52 we have

Corollary 3.53 *Suppose $(A_n) \subset L(E)$, $A \in S(E)$. Then*

$$A_n \xrightarrow{\mathcal{P}} A \implies A_n \xrightarrow{s} A. \quad (3.30)$$

Let us say that a set $\mathcal{A} \subset L(E)$ is *s-sequentially equicontinuous* if

$$(A_n) \subset \mathcal{A}, u_n \xrightarrow{s} 0 \implies A_n u_n \xrightarrow{s} 0.$$

Clearly, if \mathcal{A} is *s-sequentially equicontinuous*, then $\mathcal{A} \subset S(E)$. The significance here of this definition is the following result taken from [46].

Lemma 3.54 *Suppose $(A_n) \subset S(E)$, $A \in S(E)$. Then $A_n \xrightarrow{s} A$ iff $A_n \xrightarrow{S} A$ and the set $\{A_n : n \in \mathbb{N}\}$ is *s-sequentially equicontinuous*.*

Let us say that a set $\mathcal{A} \subset L(E)$ is *S-sequentially compact* if, for every sequence $(A_n) \subset \mathcal{A}$, there exists $A \in \mathcal{A}$ and a subsequence (A_{n_m}) such that $A_{n_m} \xrightarrow{S} A$. Then Lemma 3.54 and other observations made above imply the following corollary.

Corollary 3.55 *Suppose $\mathcal{A} \subset L(E)$. Then \mathcal{A} is *s-sequentially compact* iff $\mathcal{A} \subset S(E)$ and \mathcal{A} is *s-sequentially equicontinuous* and *S-sequentially compact*.*

Although (E, s) is not metrisable, there are versions of the Banach-Steinhaus theorem [157, 151] that would apply if (E, s) were a Baire space or, more generally, a barrelled TVS, to give that $\{A_n : n \in \mathbb{N}\}$ is *s-sequentially equicontinuous* if $A_n \xrightarrow{S} A$ and $(A_n) \subset S(E)$. But, by [46, Theorem 2.1], (E, s) is not barrelled. In fact the following example makes it clear that a version of the Banach-Steinhaus theorem, enabling equicontinuity to be deduced from continuity and pointwise boundedness, does not always hold for (E, s) .

Example 3.56 Let $E = \ell^\infty(\mathbb{Z}, \mathbb{C})$. For $n \in \mathbb{N}$ define $A_n \in L(E)$ by $(A_n u)(m) = u(n)$, for $u \in E$, $m \in \mathbb{Z}$. It is easy to see that $(A_n) \subset S(E) \subset L(E)$, and clearly $\|A_n\| \leq 1$ so that (A_n) is bounded. But (A_n) is not *s-sequentially equicontinuous* as, defining $u_n(m) = 1 + \tanh(m - n)$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$, clearly $(u_n) \subset E$, $u_n \xrightarrow{s} 0$, but $(A_n u_n)(0) = 1$ for all n , so $A_n u_n \not\xrightarrow{s} 0$. □

In the case that E satisfies an additional assumption, it is shown in [46] that a sequence $(A_n) \subset S(E)$ that is S -convergent is s -sequentially equicontinuous. The additional assumption, called ‘Assumption A’ in [46, 39], applies to $E = E^p(X)$ if $p = \infty$.

Lemma 3.57 [46] *Suppose that $p = \infty$ and that $(A_n) \subset S(E)$, $A \in S(E)$, and $A_n \xrightarrow{S} A$. Then $\{A_n : n \in \mathbb{N}\}$ is s -sequentially equicontinuous.*

Combining Lemmas 3.57, 3.54, Corollary 3.53, and (3.28), we have the following result which shows that, when $p = \infty$, the convergence \xrightarrow{S} is weaker than both ordinary strong convergence and \mathcal{P} -convergence.

Corollary 3.58 *Suppose that $p = \infty$ and that $(A_n) \subset S(E)$, $A \in S(E)$. Then*

$$A_n \xrightarrow{\mathcal{P}} A \quad \Rightarrow \quad A_n \xrightarrow{s} A \quad \Leftrightarrow \quad A_n \xrightarrow{S} A \quad \Leftarrow \quad A_n \rightarrow A.$$

3.6 Infinite Matrices

Let $E = E^p(X) = \ell^p(\mathbb{Z}^N, X)$ as before, where $p \in \{0\} \cup [1, \infty]$, N is a natural number and X a complex Banach space.

3.6.1 Inducing Matrix vs. Representation Matrix

Given an infinite matrix $M = (m_{ij})_{i,j \in \mathbb{Z}^N}$ with operator entries $m_{ij} \in L(X)$, we will say that M induces the operator

$$(Au)(i) = \sum_{j \in \mathbb{Z}^N} m_{ij} u(j), \quad i \in \mathbb{Z}^N \quad (3.31)$$

on E if the sum converges in X for every $i \in \mathbb{Z}^N$ and every $u = ((u(j)))_{j \in \mathbb{Z}^N} \in E$ and if the resulting operator A is a bounded mapping $E \rightarrow E$.

Conversely, to every operator $A \in L(E)$ one can associate an infinite matrix $[A] = (a_{ij})_{i,j \in \mathbb{Z}^N}$ by the following construction. For $k \in \mathbb{Z}^N$, let $E_k : X \rightarrow E$ and $R_k : E \rightarrow X$ be extension and restriction operators, defined by $E_k x = (\dots, 0, x, 0, \dots)$, for $x \in X$, with the x standing at the k th place in the sequence, and by $R_k u = u(k)$, for $u = (u(j))_{j \in \mathbb{Z}^N} \in E$. Then the matrix entries of $[A]$ are defined as

$$a_{ij} := R_i A E_j \in L(X), \quad i, j \in \mathbb{Z}^N, \quad (3.32)$$

and $[A]$ is called the *matrix representation* of A .

A straightforward computation shows that if M is an infinite matrix and A is induced, via (3.31), by M then the matrix representation $[A]$ from (3.32) is equal to M . It does not work quite like that the other way round:

Start with an operator $A \in L(E)$, put $M := [A]$ with entries (3.32) and ask whether or not M induces the same operator A via (3.31). Sometimes the answer will be ‘no’. But first note that, for every $u \in E_{00} = \cup_n \text{im } P_n$, the i th component of Au is given by (3.31) with $m_{ij} = a_{ij}$ from (3.32) so that $[A] = (a_{ij})$ uniquely determines Au for all $u \in E_0 = \text{clos } E_{00}$. Consequently, the restricted operator $A_0 := A|_{E_0}$ is uniquely determined by $[A]$; and of course this is also true the other way round: A_0 uniquely determines (3.32) for all $i, j \in \mathbb{Z}^N$ and hence $[A]$,

$$[A] \longleftrightarrow A_0.$$

Now if $p \in \{0\} \cup [1, \infty)$ then $E = E_0$ and hence $A = A_0$ is uniquely determined by its matrix representation $[A]$.

For $p = \infty$, there are however operators $A \in L(E)$ (see e.g. Example 3.12.c, where $[A] = 0$) for which the matrix representation $M := [A]$ induces an operator that is different from A . Under additional conditions on the operator A this problem can be overcome also in case $p = \infty$. For example, if $A \in S(E)$ then A is uniquely determined by A_0 , and hence by $[A]$, via the extension formula (3.8) and Lemma 3.15. One can show that the same argument still applies under the assumption that $P_m A Q_n \rightarrow 0$ as $n \rightarrow \infty$ for each $m \in \mathbb{N}$ (instead of $P_m A Q_n \rightrightarrows 0$, i.e. $A \in S(E)$). Another framework in which A is uniquely determined by $[A] = (a_{ij})$ in case $p = \infty$ is when X has a predual X^\triangleleft and $A \in L^\triangleleft(E)$ since then A^\triangleleft on $E^1(X^\triangleleft)$ is uniquely determined by its matrix representation (a_{ji}^\triangleleft) and $A = (A^\triangleleft)^*$.

Although the entries of our infinite matrices M are indexed slightly more complicated than in the matrices that one usually has in mind, we will speak about rows, columns and diagonals of $M = (m_{ij})_{i,j \in \mathbb{Z}^N}$ in the usual way: We refer to the sequence $(m_{ij})_{j \in \mathbb{Z}^N}$ as the i th row, to $(m_{ij})_{i \in \mathbb{Z}^N}$ as the j th column, and to $(m_{i,i-k})_{i \in \mathbb{Z}^N}$ as the k th diagonal of M – or simply, of the operator A induced by M via (3.31). As usual, the 0th diagonal is also called the *main diagonal*. Moreover, we will refer to $(m_{i,-i})_{i \in \mathbb{Z}^N}$ as the *cross diagonal* of M .

3.6.2 Our Operator Classes from the Matrix Point of View

Many of the operator classes that we have introduced already can be nicely characterised in terms of the matrix pattern of their members. To see this, suppose

$A \in L(E)$ is induced by an infinite matrix $M = (m_{ij})_{i,j \in \mathbb{Z}^N}$ with operator entries $m_{ij} \in L(X)$. Then we will say that

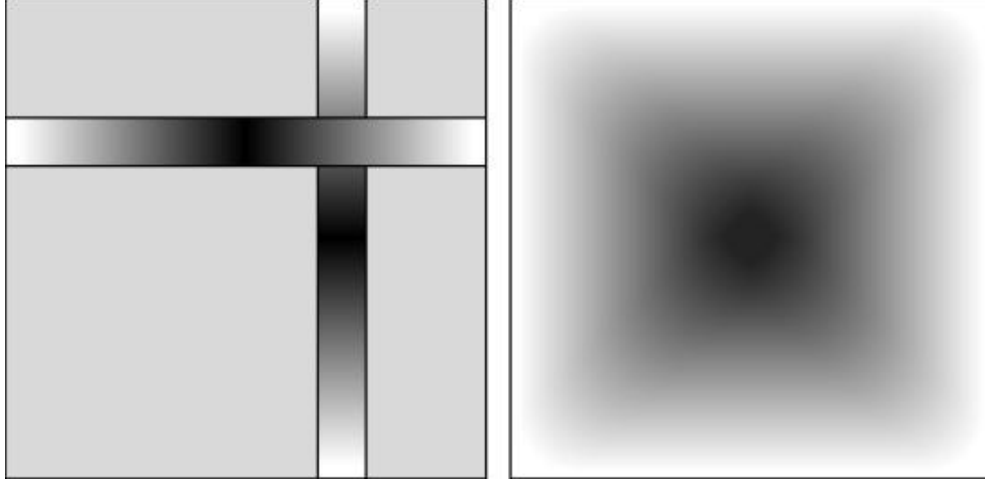


Figure 3.3: Decay properties of infinite matrices that correspond to operators in $L(E, \mathcal{P})$ (left) and $K(E, \mathcal{P})$ (right).

$A \in S^0(E)$ if every row of M has finitely many nonzero entries only;

$A \in L^0(E, \mathcal{P})$ if every row and every column of M have finitely many nonzero entries only;

$A \in SN^0(E)$ if M is supported in finitely many columns only;

$A \in K^0(E, \mathcal{P})$ if M is supported in finitely many rows and columns (i.e. has finitely many nonzero entries at all) only;

$A \in BO(E)$ if M is supported in finitely many diagonals only.

It is clear that

$$K^0(E, \mathcal{P}) \subset BO(E) \subset L^0(E, \mathcal{P}) \subset S^0(E)$$

and

$$K^0(E, \mathcal{P}) \subset SN^0(E) \subset S^0(E)$$

hold. Moreover, it is easy to check that all of them are subalgebras (none of them closed) of $L(E)$, and that $K^0(E, \mathcal{P})$ is a twosided ideal in $L^0(E, \mathcal{P})$ and a left-ideal

in $S^0(E)$, whereas $SN^0(E)$ is a twosided ideal in $S^0(E)$ and still a right-ideal in $L(E)$. More precisely, $L^0(E, \mathcal{P})$ is the idealiser of $K^0(E, \mathcal{P})$ (i.e. the set of all $A \in L(E)$ for which both AK and KA are in $K^0(E, \mathcal{P})$ whenever $K \in K^0(E, \mathcal{P})$) and $S^0(E)$ is the left-idealiser (all $A \in L(E)$ for which $KA \in SN^0(E)$ whenever $K \in SN^0(E)$) and therefore idealiser of $SN^0(E)$ (note that $AK \in SN^0(E)$ for all $A \in L(E)$ and $K \in SN^0(E)$).

All of these properties extend to the closures of $S^0(E)$, $L^0(E, \mathcal{P})$, $SN^0(E)$, $K^0(E, \mathcal{P})$ and $BO(E)$ in the norm of $L(E)$ which are, respectively², our classes $S(E)$, $L(E, \mathcal{P})$, $SN(E)$, $K(E, \mathcal{P})$ from Section 3.2 and the set $BDO(E)$ from Section 3.4. So in particular, the inclusions

$$K(E, \mathcal{P}) \subset BDO(E) \subset L(E, \mathcal{P}) \subset S(E)$$

and

$$K(E, \mathcal{P}) \subset SN(E) \subset S(E)$$

hold, all sets are Banach subalgebras of $L(E)$, and the same ideal/idealiser³ relations hold as above.

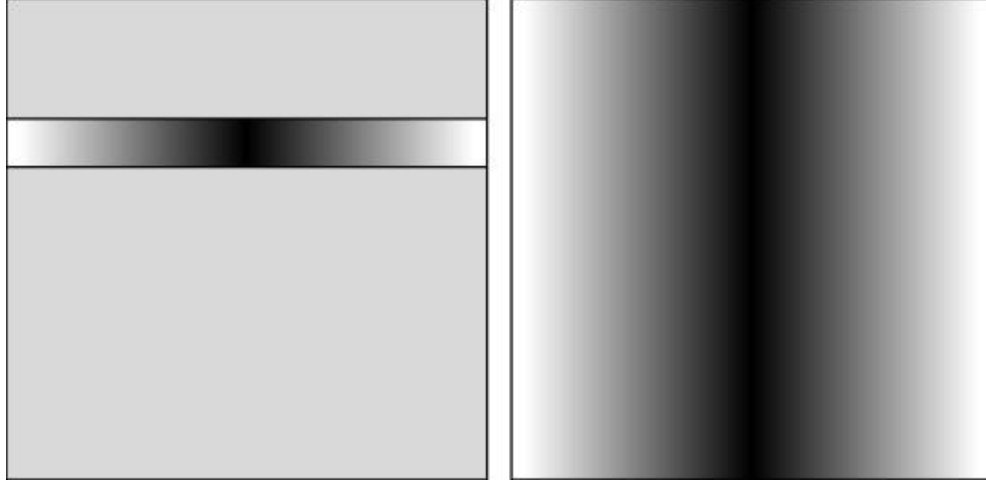


Figure 3.4: Decay properties of infinite matrices that correspond to operators in $S(E)$ (left) and $SN(E)$ (right).

The matrix entries in each row, resp. row and column, decay at infinity if $A \in S(E)$, resp. $L(E, \mathcal{P})$, whereas the same behaviour happens in a uniform way if $A \in SN(E)$, resp. $A \in K(E, \mathcal{P})$ (see Figures 3.3 and 3.4).

²The proof of this is a simple exercise.

³Note that, in general, the idealiser of the closure need not be equal to the closure of the idealiser. In our examples however, they are equal.

In contrast to $S(E)$, $L(E, \mathcal{P})$, $SN(E)$ and $K(E, \mathcal{P})$, membership in the classes $M(E)$ and $KS(E)$, also introduced in Section 3.2, cannot be read off $[A]$ in terms of decay properties of $\|a_{ij}\|$ as i and j go to infinity in certain ways. Instead, it depends on compactness properties of every single matrix entry a_{ij} as an operator on X . For example, it is easy to see that an operator $A \in S(E)$ is in $M(E)$ iff all its entries a_{ij} are in $K(X)$. We will come back to these and related questions later in Chapter 5 (see e.g. Lemma 5.2).

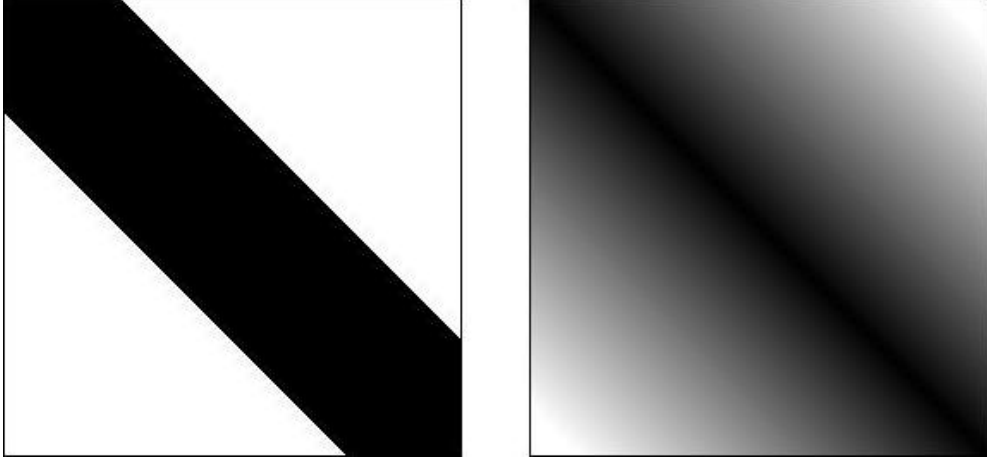


Figure 3.5: Decay properties of infinite matrices that correspond to operators in $BO(E)$ (left) and $BDO(E)$ (right).

As band- and band-dominated operators play an especially prominent role in what follows, they have their own section now.

3.7 Band- and Band-Dominated Operators

For $A \in BDO(E) \subset S(E)$, the matrix M that induces A via (3.31) and the matrix representation $[A]$ from (3.32) are the same; so it is safe to speak about matrix entries and diagonals of the operator A if $A \in BDO(E)$.

3.7.1 Measures of Off-Diagonal Decay

For a classification of band-dominated operators, it is helpful to define, for every $A \in BDO(E)$ and $k \in \mathbb{Z}^N$,

$$d_k(A) := (a_{i, i-k})_{i \in \mathbb{Z}^N} \quad \text{and} \quad d(A) := (d_k(A))_{k \in \mathbb{Z}^N}$$

as a shorthand for the k th diagonal of A and the sequence of those, and

$$\delta_k(A) := \|d_k(A)\|_\infty \quad \text{and} \quad \delta(A) := (\delta_k(A))_{k \in \mathbb{Z}^N}$$

as the k th diagonal's supremum norm (note that, necessarily for $A \in L(E)$, every diagonal of A is a bounded sequence) and the sequence of these norms.

Remark 3.59 It should be noted that, for every $k \in \mathbb{Z}^N$, the function $b_k : \mathbb{Z}^N \rightarrow L(X)$ from (3.20), thought of as a sequence over \mathbb{Z}^N with values in $L(X)$, coincides with the k th diagonal $d_k(A)$ introduced above. \square

Clearly, $A \in BO(E)$ iff $d(A) \in c_{00}(\mathbb{Z}^N, Y)$ with $Y = \ell^\infty(\mathbb{Z}^N, L(X))$; that is, $\delta(A)$ has finitely many nonzero entries only. Recall that the smallest $w \in \mathbb{N}_0$ for which $\delta_k(A) = 0$ if $|k| > w$ is the so-called bandwidth of A . In order to distinguish between band operators of different bandwidths, we introduce the notation $BO_w(E)$ for the set of all $A \in BO(E)$ whose matrix is supported on the diagonals $d_k(A)$ with $k \in \{-w, \dots, w\}^N$ only, i.e. $\delta_k(A) = 0$ if $|k| > w$. Note that $BO_w(E)$ is merely a linear space – it is neither closed nor an algebra. In the particular case when $N = 1$ and $w = 1$, operators in $BO_w(E)$ are referred to as *Jacobi operators* and their matrix representations are called tridiagonal (or Jacobi) matrices.

We have

$$BO_w(E) \subset BO_{w+1}(E) \quad \text{and} \quad BO(E) = \bigcup_{w=0}^{\infty} BO_w(E),$$

and $A \in BDO(E)$ iff

$$0 = \text{dist}(A, BO(E)) = \text{dist}\left(A, \bigcup_{w=0}^{\infty} BO_w(E)\right) = \lim_{w \rightarrow \infty} \text{dist}(A, BO_w(E))$$

with the usual definition of the distance, $\text{dist}(A, S) := \inf_{B \in S} \|A - B\|$, of an operator $A \in L(E)$ from a set $S \subset L(E)$. Note that if a_{ij} is a matrix entry of A with $|i - j| > w$ then clearly a_{ij} is still a matrix entry of $A - B$ for all $B \in BO_w(E)$ so that $\|A - B\| \geq \|a_{ij}\|$. Consequently,

$$\text{dist}(A, BO_w(E)) = \inf_{B \in BO_w(E)} \|A - B\| \geq \|a_{ij}\|, \quad |i - j| > w$$

holds, i.e. the number $\text{dist}(A, BO_w(E))$ is a bound on all matrix entries outside the $\{-w, \dots, w\}^N$ band of $[A]$. Using the diagonal suprema $\delta_k(A)$ defined above, we can rephrase this as

$$\begin{aligned} \text{dist}(A, BO_w(E)) &\geq \sup\{\|a_{ij}\| : i, j \in \mathbb{Z}^N, |i - j| > w\} \\ &= \delta_k(A), \quad \text{for all } |k| > w. \end{aligned} \tag{3.33}$$

Consequently, $\delta(A) \in c_0(\mathbb{Z}^N, \mathbb{C})$ if $A \in BDO(E)$. Note that the fact that $A \in BDO(E)$ does not imply a particular decay rate of the diagonal norms $\delta_k(A)$ as $|k| \rightarrow \infty$. In fact, for every positive null sequence z_0, z_1, \dots , there is an operator $A \in BDO(E)$ such that $\delta_k(A) = z_{|k|}$ for all $k \in \mathbb{Z}^N$. For example, take $(Au)(i) = z_{|i|}u(-i)$ for all $i \in \mathbb{Z}^N$. Then $[A]$ is only supported on the cross diagonal where $a_{i,-i} = z_{|i|}I_X \rightrightarrows 0$ as $|i| \rightarrow \infty$. We will see below that, although no particular decay rate of $\delta_k(A)$ is necessary for $A \in BDO(E)$, a reasonably fast decay is sufficient for it.

3.7.2 Characterisations of $BO(E)$ and $BDO(E)$

We start with a simple characterisation (see Proposition 1.36 in [106]) of band operators.

Proposition 3.60 *For a bounded linear operator A on $E = E^p(X)$, the following are equivalent:*

- (i) A is a band operator with bandwidth w ,
- (ii) $P_V A P_U = 0$ for all sets $U, V \subset \mathbb{Z}^N$ with $\text{dist}(U, V) > w$.

The corresponding result for band-dominated operators (see Theorem 1.42 in [106]) is this:

Proposition 3.61 *For a bounded linear operator A on $E = E^p(X)$, the following are equivalent:*

- (i) A is a band-dominated operator,
- (ii) $P_V A P_U \rightrightarrows 0$ as $\text{dist}(U, V) \rightarrow \infty$ in the following sense:
 $\forall \varepsilon > 0 \exists d > 0 : \|P_V A P_U\| < \varepsilon$ for all $U, V \subset \mathbb{Z}^N$ with $\text{dist}(U, V) > d$.

The above characterisations show (and we already mentioned) that $BO(E^p)$ does not, but $BDO(E^p)$ does, depend on the value of p . Here is an example of the latter dependence.

Example 3.62 – Laurent Operators. We have a short intermezzo on Laurent operators. These are operators, the matrix representation of which is constant along every diagonal. Let $p = 2$, $N = 1$, $X = \mathbb{C}$, and fix a function

$a \in L^\infty(\mathbb{T})$, where \mathbb{T} is the unit circle in the complex plane. The operator

$$A = \sum_{k=-\infty}^{\infty} a_k V_k \quad \text{i.e.} \quad [A] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

on $E = E^2 = \ell^2(\mathbb{Z}, \mathbb{C})$ is called *Laurent operator*, where $a_k \in \mathbb{C}$ are the Fourier coefficients

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z}$$

of the function a on the unit circle \mathbb{T} , which is referred to as the *symbol* of $A =: L(a)$.

For $b \in L^2(\mathbb{T})$, let $\hat{b} = (b_n)_{n \in \mathbb{Z}} \in E^2$ denote its sequence of Fourier coefficients and note that $L(a)\hat{b} = \hat{a} \star \hat{b} = \widehat{ab}$ acts as the operator of convolution by the sequence $\hat{a} = (a_n)$. In other words, if $F : L^2(\mathbb{T}) \rightarrow E^2$ is the Fourier transform $b \mapsto \hat{b} = (b_n)$ then

$$L(a) = F M(a) F^{-1}$$

with $M(a) : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ denoting the multiplication operator $b \mapsto ab$. As a consequence, one gets that

$$L(a) \in L(E^2) \iff a \in L^\infty(\mathbb{T}),$$

$$\|L(a)\| = \|a\|_\infty, \quad a \in L^\infty(\mathbb{T}), \quad (3.34)$$

$$L(a) + L(b) = L(a + b), \quad a, b \in L^\infty(\mathbb{T}), \quad (3.35)$$

$$L(a)L(b) = L(ab), \quad a, b \in L^\infty(\mathbb{T}),$$

$$L(a) \text{ is invertible in } L(E^2) \iff a \text{ is invertible in } L^\infty(\mathbb{T}).$$

As a consequence of (3.34) and (3.35), we moreover get that

$$L(a) \in BO(E^2) \iff \#\{k : a_k \neq 0\} < \infty \iff a \text{ is trig. polynomial},$$

$$L(a) \in BDO(E^2) \iff a \in \text{clos}_{L^\infty(\mathbb{T})}\{\text{trigonometric polynomials}\} = C(\mathbb{T}).$$

Note that the class M^p of all symbols $a \in L^\infty(\mathbb{T})$, for which $L(a) \in L(E^p)$ holds, decreases as soon as $p \in [1, \infty]$ differs from 2. The same is true for the set of all $a \in M^p$ for which $L(a) \in BDO(E^p)$. For every $p \in [1, \infty]$, the set M^p is a Banach subalgebra of $M^2 = L^\infty(\mathbb{T})$, equipped with the norm

$$\|a\|_{M^p} := \|L(a)\|_{L(\ell^p)}. \quad (3.36)$$

Moreover, it holds that

$$M^1 \subset M^{p_1} \subset M^{p_2} \subset M^2 = L^\infty(\mathbb{T})$$

if $1 \leq p_1 \leq p_2 \leq 2$, and $M^p = M^q$ if $1/p + 1/q = 1$, including the equality $M^1 = M^\infty$. From (3.36) it follows that

$$L(a) \in BDO(E^p) \iff a \in \text{clos}_{M^p}\{\text{trigonometric polynomials}\} \subset C(\mathbb{T}).$$

It follows that $L(a) \in BDO(E^{p_1})$ implies $L(a) \in BDO(E^{p_2})$ for all $1 \leq p_1 \leq p_2 \leq 2$, and that $L(a) \in BDO(E^p)$ if and only if $L(a) \in BDO(E^q)$ with $1/p + 1/q = 1$. In particular, the latter holds with $p = 1$ and $q = \infty$ if and only if $a \in W(\mathbb{T})$; that is, $\sum |a_k| < \infty$, which is a proper subclass of $C(\mathbb{T})$. \square

Remark 3.63 The impression that, for every $A \in BDO(E)$, it holds that $A_m \rightrightarrows A$ where $[A_m]$ is just the restriction of $[A]$ to a finite number of diagonals, is false in general!

As a counter-example, take $p = 2$, $N = 1$, $X = \mathbb{C}$, and $A = L(a)$, where $a \in C(\mathbb{T})$ is such that the sequence of partial Fourier sums

$$t \in \mathbb{T} \mapsto \sum_{k=-m}^m a_k t^k$$

is not uniformly convergent to a as $m \rightarrow \infty$. Consequently, by (3.34),

$$A_m = \sum_{k=-m}^m a_k V_k \not\rightarrow A \quad \text{as } m \rightarrow \infty.$$

By Fejer's theorem [87, Theorem 3.1] we know that, however, the Fejer Cesaro means uniformly converge to a ; that is, the functions

$$t \mapsto \sum_{k=-m}^m \left(1 - \frac{|k|}{m+1}\right) a_k t^k$$

converge to a in the norm of $L^\infty(\mathbb{T})$ as $m \rightarrow \infty$, showing that $\tilde{A}_m \rightrightarrows A$ with

$$\tilde{A}_m = \sum_{k=-m}^m \left(1 - \frac{|k|}{m+1}\right) a_k V_k \in BO(E).$$

\square

3.7.3 The Wiener Algebra

It is possible, and helpful, to single out subclasses of $BDO(E)$ by restricting oneself to a particular decay rate of the sequence $\delta(A)$. Examples of such decay rates (e.g. [95]) are (apart from $\delta_k(A)$ being eventually zero, i.e. band operators) exponential decay or absolutely summable decay.

In the latter case, one imposes that $\delta(A) \in \ell^1(\mathbb{Z}^N, \mathbb{C})$; that is, $d(A) \in \ell^1(\mathbb{Z}^N, Y)$ with $Y = \ell^\infty(\mathbb{Z}^N, L(X))$, i.e.

$$\|A\|_{\mathcal{W}} = \sum_{k \in \mathbb{Z}^N} \delta_k(A) = \sum_{k \in \mathbb{Z}^N} \|d_k(A)\|_\infty = \sum_{k \in \mathbb{Z}^N} \sup_{j \in \mathbb{Z}^N} \|a_{j+k,j}\| < \infty,$$

where $\|A\|_{\mathcal{W}}$ was first defined in (3.21). So this class of operators coincides with the class $\mathcal{W}(E)$ that we introduced earlier and is indeed a subset of the band-dominated operators. From

$$d(AB) = d(A) \star d(B) \quad \text{with} \quad (f_k) \star (g_k) = \left(\sum_i f_{k-i} g_i \right)_k,$$

the fact that $Y = \ell^\infty(\mathbb{Z}^N, L(X))$ is a Banach algebra under pointwise addition and multiplication and that $\ell^1(\mathbb{Z}^N, Y)$ is a Banach algebra under pointwise addition and convolution \star , we get that \mathcal{W} is a Banach algebra – the so-called *Wiener algebra*.

Remark 3.64 Suppose $A \in \mathcal{W}(E)$, and $[A] = (a_{ij})_{i,j \in \mathbb{Z}^N}$. Unlike for arbitrary band-dominated operators A (see Remark 3.63), for operators $A \in \mathcal{W}(E)$, the sequence of band operators A_m with $[A_m] = (a_{ij})_{|i-j| \leq m}$ converges to A in the $\|\cdot\|_{\mathcal{W}}$ norm, and hence in the norm of $L(E)$ for all $p \in \{0\} \cup [1, \infty]$. \square

The Classical Wiener Algebra and Wiener's Theorem

The term 'Wiener algebra' is commonly used for the set $W(\mathbb{T})$ of all functions $f(t) = \sum_{n \in \mathbb{Z}} f_n t^n$ on the complex unit circle \mathbb{T} whose sequence of Fourier coefficients $\hat{f} = (f_n)_{n \in \mathbb{Z}}$ is in $\ell^1(\mathbb{Z}, \mathbb{C})$, equipped with pointwise addition and multiplication and with the norm $\|f\| := \sum |f_n|$. Norbert Wiener's famous theorem says that if $f \in W(\mathbb{T})$ is invertible as a continuous function, i.e. f vanishes nowhere on \mathbb{T} , then $f^{-1} = 1/f$ is in $W(\mathbb{T})$ as well, showing that $W(\mathbb{T})$ is inverse closed. In fact, Wiener's theorem is a special case of a very deep result, Proposition 3.65 below, about our set $\mathcal{W}(E)$ of operators being inverse closed. It follows if we identify $W(\mathbb{T})$ with the subalgebra of all Laurent operators in $\mathcal{W}(E)$ by identifying $f \in W(\mathbb{T})$ with $L(f)$ from Example 3.62 and then apply Proposition 3.65 c) to $L(f)$.

To see this, take $E = E^2 = \ell^2(\mathbb{Z}, \mathbb{C})$ and associate with every function $f \in W(\mathbb{T})$ the Laurent operator $A = L(f)$. Then

$$L(f) \in \mathcal{W}(E) \quad \text{and} \quad \|L(f)\|_{\mathcal{W}} = \sum_{n \in \mathbb{Z}} |f_n| = \|f\|.$$

Now recall that $L(f) = F M(f) F^{-1}$, where $F : L^2(\mathbb{T}) \rightarrow E^2$ is the Fourier transform and $M(f) : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ is the multiplication operator $g \mapsto fg$. So $L(f)$ is invertible on E^2 iff $M(f)$ is invertible on $L^2(\mathbb{T})$ which is clearly the case iff f has no zeros on \mathbb{T} . In this case

$$(L(f))^{-1} = F M(f^{-1}) F^{-1} = L(f^{-1})$$

holds, and Wiener's statement, $f^{-1} \in W(\mathbb{T})$, is equivalent to the inverse Laurent matrix $(L(f))^{-1} = L(f^{-1})$ being in $\mathcal{W}(E)$. Our proposition however says much more: For all operators $A \in \mathcal{W}(E)$, not just for those with constant diagonals, the inverse A^{-1} , if it exists, is in $\mathcal{W}(E)$.

Proposition 3.65 a) $L(E, \mathcal{P})$ is inverse closed in $L(E)$.

b) $BDO(E)$ is inverse closed in $L(E)$.

c) $\mathcal{W}(E)$ is inverse closed in $L(E)$.

Proof. **a)** is from [143, Theorem 1.1.9]. For **b)** see [143, Proposition 2.1.8] or [106, Proposition 1.46]. A full proof of **c)**, that mostly goes back to [95], is in [143, Theorem 2.5.2]. The proof of the classic result with constant diagonals is in [17]. ■

Invertibility and Fredholmness are Independent of p

Recall that an operator $A \in \mathcal{W}(E)$ acts as a bounded (in fact band-dominated) operator on all spaces $E = E^p$ with $p \in \{0\} \cup [1, \infty]$. This is why we often just write \mathcal{W} in place of $\mathcal{W}(E)$.

One corollary of Proposition 3.65 c) is that if an operator $A \in \mathcal{W}$ is invertible on a space E^p then its inverse is again given by an operator in \mathcal{W} and A^{-1} therefore acts boundedly on all spaces E^p with $p \in \{0\} \cup [1, \infty]$.

Corollary 3.66 *Invertibility and hence spectrum of $A \in \mathcal{W}(E^p)$ do not depend on p .*

A similar statement was shown in [107] for Fredholmness, the Fredholm index and the essential spectrum of $A \in \mathcal{W}(E^p)$. We now state and prove this result,

where, for a Banach space X , we write $X \in \mathcal{H}$ if X is finite-dimensional or if it possesses a subspace of codimension 1 that is isomorphic to X . Now let $E = E^p = E^p(X)$.

Theorem 3.67 *If $X \in \mathcal{H}$ and $A \in \mathcal{W}$ is Fredholm on one of the spaces E^p with $p \in \{0\} \cup [1, \infty]$ (existence of predual X^Δ and preadjoint A^Δ assumed if $p = \infty$) then its Fredholm regularizer $B \in L(E^p)$ can be chosen in \mathcal{W} as well, and the remainders $AB - I$ and $BA - I$ are compact on all spaces E^p with $p \in \{0\} \cup [1, \infty]$.*

Theorem 3.68 *Let $X \in \mathcal{H}$ and $A \in \mathcal{W}$. Then the following hold.*

- a) *If A is Fredholm on one of the spaces E^p with $p \in \{0\} \cup [1, \infty)$ then A is Fredholm on all the spaces E^q with $q \in \{0\} \cup [1, \infty]$.*
- b) *If X has a predual X^Δ and A , considered as acting on $E^\infty = \ell^\infty(\mathbb{Z}^N, X)$, has a preadjoint A^Δ on $\ell^1(\mathbb{Z}^N, X^\Delta)$ and if A is Fredholm on E^∞ then A is Fredholm on all the spaces E^q with $q \in \{0\} \cup [1, \infty]$.*

In both cases, the index of A is the same on all these spaces E^q with $q \in \{0\} \cup [1, \infty]$.

We now come to the proof, that is largely motivated by Roch's paper [152], of these two theorems. We start with two basic lemmas ([108, Lemma 2.1 and 2.2]).

Lemma 3.69 *$A \in L(E)$ is Fredholm of index zero iff there exist an invertible operator $B \in L(E)$ and a compact operator $K \in K(E)$ such that $A = B + K$.*

Lemma 3.70 *The following are equivalent for an infinite-dimensional complex Banach space X .*

- (i) *X is isomorphic to a subspace $Y \subset X$ of codimension 1.*
- (ii) *X is isomorphic to $X \times \mathbb{C} = \{(x, \lambda) : x \in X, \lambda \in \mathbb{C}\}$.*
- (iii) *There exists a Fredholm operator $S \in L(X)$ with $\text{ind } S = 1$.*
- (iv) *There exists a Fredholm operator $T \in L(X)$ with $\text{ind } T = -1$.*

It has been an open problem, the so-called *hyperplane problem*, posed by Stefan Banach in the famous “Scottish Book”, whether or not there are any complex Banach spaces outside of \mathcal{H} . In 1993, more than 50 years later, William Timothy Gowers [77] solved this and two more of Banach's classical problems by constructing a Banach space that is not in \mathcal{H} . Gowers was subsequently awarded the Fields Medal in 1998 for his contributions to functional analysis by combining it with combinatorial ideas. Further examples $X \notin \mathcal{H}$ were given by Koszmider [89] and Plebanek [130]. Note that all three authors constructed Banach spaces X

for which **no** subspace of finite codimension is isomorphic to X . As a consequence, in $L(X)$ there are no Fredholm operators with a non-zero index!

We here study Fredholm operators on spaces of functions $\mathbb{Z}^N \rightarrow X$ with $X \in \mathcal{H}$. Judging by the fact that the discovery of Banach spaces $X \notin \mathcal{H}$ took a long time (and was worth a Fields Medal) it seems pretty safe to assume that your given Banach space X at hand is contained in \mathcal{H} and is therefore covered by Theorems 3.67 and 3.68. We will however give some sufficient criteria here for membership in \mathcal{H} . The following lemma (Lemma 2.5 in [108]) is the result of communication with Les Bunce from Reading, UK.

Lemma 3.71 *Let X be an infinite-dimensional complex Banach space. Then the following hold.*

- (i) $X \in \mathcal{H}$ implies that $X^* \in \mathcal{H}$. The converse is in general not true.
- (ii) The direct sum $X = Y \dot{+} Z$ is in \mathcal{H} if one of Y and Z is in \mathcal{H} .
- (iii) The spaces $c_0 := c_0(\mathbb{N}, \mathbb{C})$ and $\ell^p := \ell^p(\mathbb{N}, \mathbb{C})$ with $1 \leq p \leq \infty$ are in \mathcal{H} . Consequently, all spaces $c_0(\Omega, Y)$ and $\ell^p(\Omega, Y)$ with Ω at most countable, Y a finite-dimensional complex space, and $1 \leq p \leq \infty$ are in \mathcal{H} .
- (iv) If $c_0 \lesssim X$ (meaning that X contains an isomorphic copy of c_0) and X is separable, then $X \in \mathcal{H}$.
- (v) If $c_0 \lesssim X^*$, then $X \in \mathcal{H}$.
- (vi) If $\ell^\infty \lesssim X$, then $X \in \mathcal{H}$.
- (vii) If μ is a σ -finite nonatomic measure over an infinite set Ω and $1 \leq p \leq \infty$, then $L^p(\Omega, \mu) \in \mathcal{H}$.
- (viii) If K is an infinite compact metric space, then $C(K) \in \mathcal{H}$.
- (ix) If X is a separable C^* -algebra, then $X \in \mathcal{H}$. There are (non-separable) C^* -algebras $X \notin \mathcal{H}$.
- (x) If X is a C^* -algebra then $X^* \in \mathcal{H}$.
- (xi) If X is a von Neumann algebra, then both X and its (unique) predual X^\natural are in \mathcal{H} .

Remark 3.72 In connection with (viii), we would like to remark that already an infinite compact (not necessarily metrisable) space K is enough if it has a

nontrivial convergent sequence as this sequence can be used to construct a complementable copy of c_0 in $C(K)$ (see e.g. [130]).

We would also like to mention that there exist (non-separable) examples of $C(K) \notin \mathcal{H}$. For an example of a non-metrisable compact Hausdorff space K with this property see [89, 130]. \square

In the following, we write $\text{ind}_p A$ for the index of an operator $A \in \mathcal{W}$ on E^p . An essential ingredient to the proof of Theorems 3.67 and 3.68 is the following lemma.

Lemma 3.73 *If $X \in \mathcal{H}$ then there exists a family $\{S_\kappa\}_{\kappa \in \mathbb{Z}}$ of operators in \mathcal{W} with $\text{ind}_p S_\kappa = \kappa$ for all $p \in \{0\} \cup [1, \infty]$ and all $\kappa \in \mathbb{Z}$.*

Proof. Let $X \in \mathcal{H}$. Here is one way to choose this family.

If $n := \dim X < \infty$, let e_1, \dots, e_n be a basis in X , write $u \in E$ as

$$u(k_1, k_2, \dots, k_N) = \sum_{i=1}^n u_i(k_1, k_2, \dots, k_N) e_i$$

with $u_i(k_1, k_2, \dots, k_N) \in \mathbb{C}$ for all $k_1, \dots, k_N \in \mathbb{Z}$ and $i = 1, \dots, n$, and put

$$(S_{-1}u)(k_1, k_2, \dots, k_N) := \begin{cases} 0e_1 + \sum_{i=2}^n u_{i-1}(k_1, k_2, \dots, k_N)e_i, & k_1 = \dots = k_N = 0, \\ u_n(k_1 - 1, k_2, \dots, k_N)e_1 + \sum_{i=2}^n u_{i-1}(k_1, k_2, \dots, k_N)e_i, & k_1 > 0, k_2 = \dots = k_N = 0, \\ u(k_1, k_2, \dots, k_N), & \text{otherwise} \end{cases}$$

and

$$(S_1u)(k_1, k_2, \dots, k_N) := \begin{cases} \sum_{i=1}^{n-1} u_{i+1}(k_1, k_2, \dots, k_N)e_i + u_1(k_1 + 1, k_2, \dots, k_N)e_n, & k_1 \geq 0, k_2 = \dots = k_N = 0, \\ u(k_1, k_2, \dots, k_N), & \text{otherwise.} \end{cases}$$

If $\dim X = \infty$ choose $T_{-1}, T_1 \in L(X)$ with $\text{ind } T_{\pm 1} = \pm 1$, respectively, which is possible by $X \in \mathcal{H}$ and Lemma 3.70. Now, for every $u \in E$, put

$$(S_{\pm 1}u)(k) = \begin{cases} T_{\pm 1}(u(0)), & k = 0, \\ u(k), & k \neq 0, \end{cases}$$

respectively, for all $k \in \mathbb{Z}^N$, i.e. $S_{\pm 1} = \text{diag}(\dots, I_X, I_X, T_{\pm 1}, I_X, I_X, \dots)$ with $T_{\pm 1}$ at position zero.

In either case, $\dim X$ finite or infinite, now put

$$S_\kappa := \begin{cases} S_1^\kappa, & \kappa > 0, \\ I, & \kappa = 0, \\ S_{-1}^{-\kappa}, & \kappa < 0 \end{cases}$$

for all $\kappa \in \mathbb{Z}$, and it follows from $\text{ind}_p S_{\pm 1} = \pm 1$ for all $p \in \{0\} \cup [1, \infty]$ that $\text{ind}_p S_\kappa = \kappa$ for all $\kappa \in \mathbb{Z}$ and all p . Also note that, by our construction, $S_\kappa \in \mathcal{W}$ for all $\kappa \in \mathbb{Z}$. ■

Lemma 3.74 *If $p \in \{0\} \cup (1, \infty)$ and $K \in K(E^p)$ then $\|K - P_m K P_m\| \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. The claim follows from the bound

$$\begin{aligned} \|K - P_m K P_m\| &= \|P_m K Q_m + Q_m K\| \leq \|(P_m K Q_m)^*\| + \|Q_m K\| \\ &\leq \|Q_m^* K^*\| \cdot \|P_m^*\| + \|Q_m K\| \rightarrow 0 \end{aligned}$$

as $m \rightarrow \infty$ since

$$\|P_m\| \text{ remains bounded, } Q_m \rightarrow 0 \text{ and } Q_m^* \rightarrow 0 \text{ pointwise as } m \rightarrow \infty$$

on E^p and $(E^p)^*$, respectively, and since K and K^* are compact on E^p and $(E^p)^*$, respectively. ■

Lemma 3.75 *Let $m \in \mathbb{N}$ and $p \in \{0\} \cup [1, \infty]$. If $P_m K P_m$ is compact on E^p then it is compact on all spaces E^q with $q \in \{0\} \cup [1, \infty]$.*

Proof. Let $P_m K P_m$ be compact on E^p . Now let $q \in \{0\} \cup [1, \infty]$ and take an arbitrary bounded sequence $(u_k) \subset E^q$. We have to show that $(P_m K P_m u_k)_k$ has an E^q -convergent subsequence. W.l.o.g. we can restrict ourselves to elements $u_k \in \text{im } P_m$. Now note that on $\text{im } P_m$ all the E^q -norms are equivalent. So (u_k) is also bounded in E^p and, by our assumption, $(P_m K P_m u_k)_k$ has an E^p -convergent subsequence. But since $P_m K P_m u_k \in \text{im } P_m$ for every k and again since the norms are equivalent on $\text{im } P_m$, the same subsequence also converges in the norm of E^q . ■

We are now ready for the proof of Theorem 3.68.

Proof. Suppose $X \in \mathcal{H}$, $A \in \mathcal{W}$, $p \in \{0\} \cup [1, \infty]$ and A is Fredholm on E^p with index $\kappa := \text{ind}_p A$, and take an operator $S_{-\kappa} \in \mathcal{W}$ with $\text{ind}_p S_{-\kappa} = -\kappa$ for all $p \in \{0\} \cup [1, \infty]$ whose existence is guaranteed by Lemma 3.73.

Case 1. $p \in \{0\} \cup (1, \infty)$.

Since $AS_{-\kappa}$ is Fredholm of index zero on E^p , we know from Lemma 3.69 that there exists a compact operator K on E^p such that $AS_{-\kappa} + K$ is invertible on E^p . By Lemma 3.74 and a simple perturbation argument, we know that, for a sufficiently large $m \in \mathbb{N}$, also $A' := AS_{-\kappa} + P_m K P_m$ is invertible on E^p . Moreover,

$A' \in \mathcal{W}$ since $A, S_{-\kappa}, P_m K P_m \in \mathcal{W}$. From the inverse closedness of \mathcal{W} we know that $B' := (A')^{-1} \in \mathcal{W}$. Summarizing,

$$I = A'B' = AS_{-\kappa}B' + P_m K P_m B', \quad (3.37)$$

i.e. $AB = I - K'$ with $B = S_{-\kappa}B' \in \mathcal{W}$ and $K' = P_m K P_m B' \in \mathcal{W}$ being compact on all spaces E^q with $q \in \{0\} \cup [1, \infty]$ by Lemma 3.75. By a completely symmetric argument for $A'' := S_{-\kappa}A + P_m L P_m$ with L and m accordingly chosen, one gets that $CA = I - L'$ for some $C \in \mathcal{W}$ and $L' \in \mathcal{W}$ compact on all E^q with $q \in \{0\} \cup [1, \infty]$. Looking at $C - CK' = C(AB) = (CA)B = B - L'B$, we see that the left and right regularizers B and C only differ by an operator $L'B - CK' \in \mathcal{W}$ that is compact on all spaces E^q so that we can use one of them as regularizer for both sides. This shows that A is Fredholm on all spaces E^q . The q -independence of the index now follows by looking at (3.37) as an equality on E^q and taking the index on both sides, i.e.

$$0 = \text{ind}_q I = \text{ind}_q A + \text{ind}_q S_{-\kappa} + \text{ind}_q B' = \text{ind}_q A + (-\kappa) + 0,$$

showing that $\text{ind}_q A = \kappa = \text{ind}_p A$ for all $q \in \{0\} \cup [1, \infty]$.

Case 2. $p = \infty$ with existence of X^\natural and A^\natural .

We get from Proposition 3.37 (which is applicable since $A \in \mathcal{W}$ and since X^\natural and A^\natural exist) that A is also Fredholm, with the same index κ , if restricted to $E^0 \subset E^\infty$. Now the claim follows from Case 1 with $p = 0$.

Case 3. $p = 1$.

If A is Fredholm with index κ on $E^1 = \ell^1(\mathbb{Z}^N, X)$ then A^* is Fredholm of index $-\kappa$ on $\ell^\infty(\mathbb{Z}^N, X^*)$. By Case 2 (note that X^* and A^* clearly have a predual and preadjoint) we get that A^* is Fredholm on $\ell^2(\mathbb{Z}^N, X^*)$ with the same index $-\kappa$. But consequently, A is Fredholm on $E^2 = \ell^2(\mathbb{Z}^N, X)$ with index κ , so that the claim follows from Case 1 with $p = 2$. ■

Note that, as an interim result of this proof, we get Theorem 3.67.

3.8 Comments and References

The study of the strict topology was probably initiated by Buck [25], and the different classes of continuous and compact operators on a general Banach space with respect to this topology largely goes back to Chandler-Wilde and Zhang [46] with further modifications by Chandler-Wilde and the author [39].

\mathcal{P} -convergence of operator sequences was studied in [154], [133], [141], [143], [106] and [39], for example. Lemma 3.45 goes back to [106]. The operator convergence notions \xrightarrow{s} and \xrightarrow{S} have been introduced and studied in [46].

The classification of infinite matrices in terms of their decay properties, as presented in Section 3.6.2, can be found e.g. in [95], [143], [106] and [39]. The study of concrete classes of band and band-dominated operators (such as convolution, Wiener-Hopf, and Toeplitz operators) goes back to the 1950's starting with [91] and [73] by Gohberg and Krein and was culminating in the 1970/80's with the huge monographs [71] by Gohberg/Feldman and [21] by Böttcher and Silbermann. The study of band-dominated operators as a general operator class was initiated by Simonenko [167], [168]. More recent work along these lines can be found in [96], [140], [133] and [95], to mention some examples only. In [133] band-dominated operators are called “operators of local type” and in [95] “operators with uniformly fading memory”.

Chapter 4

Key Concepts

This chapter introduces the key concepts and develops some key results of the text. We first recall the concepts of invertibility at infinity and Fredholmness and start to explore their inter-relation. Next, we summarise some main results from the abstract generalised collectively compact operator theory developed in [46] and from the abstract theory of limit operators [140, 143, 106]. It then turns out that the collection of all limit operators of an operator A in a certain class is subject to the constraints made in the operator theory of [46]. Therefore we apply this theory and derive some new general results.

Again, let $E = E^p(X)$ be one of our sequence spaces introduced in Section 2.4 with $p \in \{0\} \cup [1, \infty]$ and a complex Banach space X .

4.1 Fredholmness and Invertibility at Infinity

4.1.1 Fredholmness Revisited

Recall from Section 2.3 that $A \in L(E)$ is called semi-Fredholm if it has a closed range and one of the numbers $\alpha(A)$ and $\beta(A)$ from (2.1) is finite, and that A is called Fredholm if both $\alpha(A)$ and $\beta(A)$ are finite (in which case its range is automatically closed). In the latter case the index of A is defined as $\alpha - \beta$. Also recall that $A \in L(E)$ is Fredholm iff there exist $B \in L(E)$ and $T_1, T_2 \in K(E)$ such that (2.3) holds. The theory of Fredholm operators is a rigorous generalization of the probably best known statement connected with the name of Erik Ivar Fredholm: *Fredholm's alternative*. In terms of (2.1), it says that, for an operator $A = I + K$ with $K \in K(E)$,

either $\alpha(A) = 0$ & $\beta(A) = 0$, or alternatively, $\alpha(A) \neq 0$ & $\beta(A) \neq 0$.

This statement is clearly true for all Fredholm operators $A \in L(E)$ with $\alpha(A) = \beta(A)$; that is $\{A \in L(E) : A \text{ Fredholm, } \text{ind} A = 0\}$. This class is strictly larger than $I + K(E)$, for which Fredholm's alternative is usually formulated. It contains all operators of the form $A = C + K$ where $C \in L(E)$ is invertible and $K \in K(E)$. Lemma 3.69 shows that it actually coincides with this class.

4.1.2 \mathcal{P} -Fredholmness and Invertibility at Infinity

For some of the things that are still to come in this text, it is convenient to replace the ideal $K(E)$ by $K(E, \mathcal{P})$ and to study the corresponding new property. Since $K(E, \mathcal{P})$ is not an ideal in $L(E)$, one gets the nicer results if one also replaces $L(E)$ by $L(E, \mathcal{P})$ which, for most applications, is not too restrictive. In analogy to Fredholmness of $A \in L(E)$, which corresponds to an invertibility problem in the Calkin algebra $L(E)/K(E)$, this naturally leads to the study of the property of an operator $A \in L(E, \mathcal{P})$ that corresponds to the invertibility of $A + K(E, \mathcal{P})$ in $L(E, \mathcal{P})/K(E, \mathcal{P})$. We will prove that this invertibility is equivalent to the existence of operators $B, C \in L(E, \mathcal{P})$ and an integer $m \in \mathbb{N}$ such that

$$Q_m AB = Q_m = CAQ_m \quad (4.1)$$

holds.

Proposition 4.1 *For all $A \in L(E, \mathcal{P})$, the following properties are equivalent:*

- (i) *The coset $A + K(E, \mathcal{P})$ is invertible in $L(E, \mathcal{P})/K(E, \mathcal{P})$.*
- (ii) *There exist $B, C \in L(E, \mathcal{P})$ with (2.2) for some $T_1, T_2 \in K(E, \mathcal{P})$.*
- (iii) *There exists a $B \in L(E, \mathcal{P})$ with (2.3) for some $T_1, T_2 \in K(E, \mathcal{P})$.*
- (iv) *There exist $B, C \in L(E, \mathcal{P})$ and an $m \in \mathbb{N}$ such that (4.1) holds.*

Proof. Obviously, (i) is equivalent to both (ii) and (iii).

(iii) \Rightarrow (iv). Take $m \in \mathbb{N}$ large enough that $\|Q_m T_1\| < 1$ and $\|T_2 Q_m\| < 1$, and put $D := (I + Q_m T_1)^{-1}$. From the Neumann series (or less elementary, from Proposition 3.65), we get that $D \in L(E, \mathcal{P})$. Then, by (2.3) and $Q_m = Q_m^2$,

$$Q_m AB = Q_m + Q_m T_1 = Q_m(I + Q_m T_1),$$

and consequently, $Q_m AB' = Q_m$ with $B' = BD \in L(E, \mathcal{P})$. The second equality in (4.1) follows from a symmetric argument using $BA = I + T_2$.

(iv) \Rightarrow (iii). If (iv) holds, then

$$AB = Q_m AB + P_m AB = Q_m + P_m AB = I - P_m + P_m AB =: I + T,$$

where $T = P_m AB - P_m \in K(E, \mathcal{P})$ since $A, B \in L(E, \mathcal{P})$. The second claim in (2.3) follows analogously. ■

Remark 4.2 The operators B and C from (ii) only differ by an operator in $K(E, \mathcal{P})$. Consequently, both can be used as the operator B in (iii). Also note that m in (4.1) clearly can be replaced by any $m' > m$. □

In accordance with [143], we call $A \in L(E, \mathcal{P})$ a \mathcal{P} -Fredholm operator if property (i) holds. Since this notion is not defined for operators outside of $L(E, \mathcal{P})$, we will also study a very similar property in the somewhat larger setting $A \in L(E)$ which enables us to compare the new property with usual Fredholmness on equal territory:

Definition 4.3 An operator $A \in L(E)$ is called invertible at infinity if there exists an operator $B \in L(E)$ such that (2.3) holds for some $T_1, T_2 \in K(E, \mathcal{P})$. Moreover, $A \in L(E)$ is called weakly invertible at infinity if there exist $B, C \in L(E)$ and an $m \in \mathbb{N}$ such that (4.1) holds.

Remark 4.4 a) Of course, A is invertible at infinity if it is invertible, as we see by putting $B = A^{-1}$ and $T_1 = T_2 = 0$ in (2.3).

b) Note that, if $A \in L(E, \mathcal{P})$ is invertible at infinity, we do not know if it is even \mathcal{P} -Fredholm, since we cannot guarantee that B from Definition 4.3 can be chosen from $L(E, \mathcal{P})$ if $A \in L(E, \mathcal{P})$. But the reverse implication is true of course: If $A \in L(E, \mathcal{P})$ is \mathcal{P} -Fredholm, then it is invertible at infinity. Moreover, we do have the following. □

If $A \in L(E)$ is invertible at infinity, then it is weakly invertible at infinity. In fact, the following slightly stronger version holds.

Lemma 4.5 If $A \in L(E)$ and if (2.3) holds with $B \in L(E)$ and $T_1, T_2 \in SN(E) \supset K(E, \mathcal{P})$, then A is weakly invertible at infinity, i.e. (4.1) holds with some other operators $B, C \in L(E)$. Moreover, if in (2.3), $B \in S(E)$ then we can also choose $B, C \in S(E)$ in (4.1). If in (2.3), $B, T_1, T_2 \in L(E, \mathcal{P})$ then also B, C from (4.1) can be chosen in $L(E, \mathcal{P})$.

Proof. Similarly to the proof of Proposition 4.1, choose $m \in \mathbb{N}_0$ large enough that $\|T_1 Q_m\| < 1$ and $\|T_2 Q_m\| < 1$. From (2.3) we get $Q_m AB Q_m = Q_m(I + T_1 Q_m)$ and $BA Q_m = (I + T_2 Q_m) Q_m$, proving that

$$Q_m A (B Q_m (I + T_1 Q_m)^{-1}) = Q_m = ((I + T_2 Q_m)^{-1} B) A Q_m,$$

i.e. (4.1) holds. The two additional claims follow immediately from the Neumann series formula and the fact that $S(E)$ and $L(E, \mathcal{P})$ are Banach subalgebras of $L(E)$. ■

We would not have chosen two different names if the two properties were equivalent. Indeed, here we give four examples of operators which are only weakly invertible at infinity.

Example 4.6 a) Let $p = 1$, $N = 1$, and let K denote the first operator in Example 3.12 a). Our example here is $A := I - K$ (note that $A := I + K$ is no good choice since this operator is even invertible with $A^{-1} = I - K/2$).

From $Q_m K = 0$ we conclude $Q_m A = Q_m$ for all $m \in \mathbb{N}_0$, and consequently, (4.1) with $B = C = Q_1$ and $m = 1$, which shows that A is weakly invertible at infinity. Concerning invertibility at infinity, note that the second equality in (2.3) holds with $B = Q_1$ and $T_2 = -P_1 \in K(E, \mathcal{P})$ for example. But we will show that the first equality in (2.3) cannot be fulfilled with $B \in L(E)$ and $T_1 \in K(E, \mathcal{P})$.

Therefore suppose that there exist such B and T_1 where $AB = I + T_1$ holds. From $Q_m B - Q_m = Q_m AB - Q_m = Q_m (AB - I) = Q_m T_1$ we conclude

$$Q_m B - Q_m \rightrightarrows 0 \quad \text{as} \quad m \rightarrow \infty.$$

Secondly, without loss of generality, we can suppose that $(Bu)_0 = 0$ for all $u \in E$ since A ignores the 0-th component of its operand. Consequently, $P_{\{0\}} B Q_m = 0$ for all $m \in \mathbb{N}$. Moreover, for all $i \in \mathbb{Z} \setminus \{0\}$,

$$P_{\{i\}} B Q_m = P_{\{i\}} A B Q_m + P_{\{i\}} K B Q_m \rightrightarrows 0 \quad \text{as} \quad m \rightarrow \infty$$

since $AB = I + T_1 \in L(E, \mathcal{P})$ and $P_{\{i\}} K = 0$ if $i \neq 0$. Choosing an arbitrary $k \in \mathbb{N}$ and summing up $P_{\{i\}} B Q_m$ over all $i = -k, \dots, k$, we get

$$P_k B Q_m \rightrightarrows 0 \quad \text{as} \quad m \rightarrow \infty.$$

Now let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ large enough that $D := Q_k - Q_k B$ has norm $\|D\| < \varepsilon/\|A\|$. Then for all $m \geq k$,

$$\begin{aligned} \|P_0 A Q_m\| &= \|P_0 A Q_k Q_m\| \leq \|P_0 A Q_k B Q_m\| + \|P_0 A D Q_m\| \\ &\leq \|P_0 A B Q_m\| + \|P_0 A P_k B Q_m\| + \|P_0 A D Q_m\| \\ &\leq \|P_0 A B Q_m\| + \|A\| \cdot \|P_k B Q_m\| + \|A\| \cdot \varepsilon/\|A\|, \end{aligned}$$

which shows that $\|P_0 A Q_m\| \rightarrow 0$ as $m \rightarrow \infty$ since $AB = I + T_1 \in L(E, \mathcal{P})$ and $\|P_k B Q_m\| \rightarrow 0$ as $m \rightarrow \infty$. Having a look at our operator A , we see that this is wrong as $\|P_0 A Q_m\| = 1$ for all $m \in \mathbb{N}$. Contradiction.

b) Again let $p = 1$ and $N = 1$. But now substitute K in a) by the second operator (denoted by \tilde{A}) of Example 3.12 a). The rest is analogously.

c) and d) Let $p = \infty$ and consider the adjoint operators A^* of a) and b), respectively. By duality arguments we see that also A^* is only weakly invertible at infinity. \square

From the definition of a \mathcal{P} -Fredholm operator it is immediately clear that this property is robust under perturbations in $K(E, \mathcal{P})$ and under perturbations with sufficiently small norm, provided the latter are in $L(E, \mathcal{P})$. While, for invertibility at infinity in $L(E)$, we conjecture that this is not true, at least the following can be shown.

Proposition 4.7 *Weak invertibility at infinity in $L(E)$ is robust*

- a) *under perturbations in $K(E, \mathcal{P})$, and*
- b) *under small perturbations in $L(E)$.*

Proof. Suppose $A \in L(E)$, and there exist $m \in \mathbb{N}$ and $B, C \in L(E)$ such that (4.1) holds.

a) Let $T \in K(E, \mathcal{P})$, and choose $m \in \mathbb{N}$ such that, in addition to (4.1), also $\|Q_m T\| < 1/\|B\|$ holds. Then from

$$Q_m(A + T)B = Q_mAB + Q_mTB = Q_m + Q_m^2TB = Q_m(I + Q_mTB)$$

and the invertibility of $I + Q_mTB$, we get that $Q_m(A + T)B' = Q_m$ holds, where $B' = B(I + Q_mTB)^{-1}$. The second equality in (4.1) is checked in a symmetric way.

b) Let $S \in L(E)$ be subject to $\|S\| \leq 1/\|B\|$. Then from

$$Q_m(A + S)B = Q_mAB + Q_mSB = Q_m + Q_mSB = Q_m(I + SB)$$

and the invertibility of $I + SB$, we get $Q_m(A + S)B' = Q_m$ with $B' = B(I + SB)^{-1}$. Again, the second equality in (4.1) is checked in a symmetric way. ■

4.1.3 Invertibility at Infinity in $BDO(E)$

Since we are especially interested in band-dominated operators, we insert some results on the invertibility at infinity of operators in $BDO(E)$.

Proposition 4.8 *$K(E, \mathcal{P})$ is an ideal in $BDO(E)$.*

Proof. This is clear since $K(E, \mathcal{P}) \subset BDO(E) \subset L(E, \mathcal{P})$ and $K(E, \mathcal{P})$ is an ideal in $L(E, \mathcal{P})$. ■

Proposition 4.8 makes it possible to study the factor algebra $BDO(E)/K(E, \mathcal{P})$. One can show that, if $A \in BDO(E)$ is invertible at infinity, the operator B in (2.3) is automatically in $BDO(E)$ as well:

Proposition 4.9 *For $A \in BDO(E)$, the following properties are equivalent.*

- (i) A is invertible at infinity.
- (ii) $A + K(E, \mathcal{P})$ is invertible in the factor algebra $BDO(E)/K(E, \mathcal{P})$.
- (iii) A is \mathcal{P} -Fredholm.

Proof. For $E = E^p$ with $1 \leq p \leq \infty$ this is precisely Proposition 2.10 of [106] (the key idea, written down for $p = 2$, is from [141, Proposition 2.6]). The proof from [106] literally transfers to $p = 0$. ■

Corollary 4.10 $BDO(E)/K(E, \mathcal{P})$ is inverse closed in $L(E, \mathcal{P})/K(E, \mathcal{P})$.

4.1.4 Invertibility at Infinity vs. Fredholmness

We have seen in Lemma 3.11 that $K(E)$ and $K(E, \mathcal{P})$ are very closely related. As a result, Fredholmness (i.e. invertibility modulo $K(E)$) and invertibility at infinity (which is invertibility modulo $K(E, \mathcal{P})$) are closely connected as well: If $K(E) \subset K(E, \mathcal{P})$, then Fredholmness implies invertibility at infinity. Conversely, if $K(E, \mathcal{P}) \subset K(E)$, then invertibility at infinity implies Fredholmness.

The aim of this section is to study these interrelations for $E = E^p(X)$ in the six essential cases determined by p and $\dim X$. We will derive a table showing which of the two properties implies the other in which of the six cases. That table is proven to be complete by giving appropriate counter-examples for the “missing” implications.

		$\dim X < \infty$	$\dim X = \infty$	
	$p = 1$	\implies		
	$p = 0$ or $1 < p < \infty$	\iff	\Leftarrow	
	$p = \infty$	\implies		
invertibility at ∞				Fredholmness

Figure 4.1: Invertibility at infinity versus Fredholmness, depending on the space $E = E^p(X)$.

As a justification for the missing implication arrows in this table, we will give some counter-examples. But before, we should mention that the implication ‘ \Leftarrow ’ can be established, under additional conditions, more often than shown above:

Firstly, further down in Theorem 5.9 (and see Remark 5.10), we will show that, for band-dominated operators, ‘ \Leftarrow ’ also holds in the first row and, under the additional condition that a preadjoint operator exists and that A is a rich operator (as to be defined below), also in the last row of the table in Figure 4.1.

Secondly, although, in general, ‘ \Leftarrow ’ does not hold in the upper left corner of our table, the following proposition shows that it “almost” holds, in the following sense.

Proposition 4.11 *Let $p = 1$ and $\dim X < \infty$. If $A \in L(E)$ is Fredholm, then it is weakly invertible at infinity.*

Proof. For the proof of the first ‘ $=$ ’ sign in (4.1), take $B \in L(E)$ and $T \in K(E)$ with $AB = I + T$. Since $Q_m \rightarrow 0$, we have $Q_m T \rightrightarrows 0$ as $m \rightarrow \infty$. Choose $m \in \mathbb{N}$ large enough that $\|Q_m T\| < 1$ and put $D := I + Q_m T$, which is invertible then. Now

$$Q_m AB = Q_m(I + T) = Q_m + Q_m T = Q_m(I + Q_m T) = Q_m D$$

shows that $Q_m AB' = Q_m$ with $B' = BD^{-1}$.

For the second ‘ $=$ ’ sign in (4.1), we first claim that, for all sufficiently large $m \in \mathbb{N}$,

$$\ker AQ_m = \ker Q_m \tag{4.2}$$

holds. Clearly, ‘ \supset ’ holds in (4.2). Suppose ‘ \subset ’ does not hold. Then there is a sequence $(m_k)_{k=1}^\infty \subset \mathbb{N}$ tending to infinity such that, for every $k \in \mathbb{N}$, there is a $x_k \in \ker AQ_{m_k} \setminus \ker Q_{m_k}$, i.e.

$$y_k := Q_{m_k} x_k \neq 0 \quad \text{where} \quad Ay_k = AQ_{m_k} x_k = 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

But these $y_k \in \operatorname{im} Q_{m_k}$ clearly span an infinite-dimensional space, which contradicts $\dim \ker A < \infty$. Consequently, (4.2) holds.

Equality (4.2) with a sufficiently large $m \in \mathbb{N}$ shows that E decomposes into a direct sum of $\ker AQ_m = \ker Q_m = \operatorname{im} P_m$ and $\operatorname{im} Q_m$. From $E = \ker AQ_m \dot{+} \operatorname{im} Q_m$ we conclude that

$$AQ_m|_{\operatorname{im} Q_m} : \operatorname{im} Q_m \rightarrow \operatorname{im} AQ_m \tag{4.3}$$

is a bijection. Since A and Q_m are Fredholm (remember that $\dim X < \infty$), we get that AQ_m is Fredholm, and hence, that $E_1 := \operatorname{im} AQ_m$ is closed and complementable. Now let E_2 denote a complementary space of E_1 and define an operator $C \in L(E)$ which acts on E_1 as the inverse of (4.3) and is zero on E_2 . With that construction, $CAQ_m = Q_m$ holds. ■

Another improvement of Figure 4.1 is given by the following result in which $C = I$ is a possible choice since $I \in S(E)$.

Theorem 4.12 *Suppose $A = C + K$, where $C \in L(E)$ is invertible, with $C^{-1} \in S(E)$, and $K \in S(E) \cap M(E)$. Suppose further that (2.3) holds with $B \in L(E)$ and $T_1, T_2 \in SN(E)$. Then A is Fredholm.*

Proof. We have from (2.3) that $CB + KB = I + T_1$, $BC + BK = I + T_2$, so that

$$B = C^{-1}(I + T_1 - KB), \quad B = (I + T_2 - BK)C^{-1}.$$

Using the first of these equations we see that

$$AC^{-1}(I - KB) = (I + KC^{-1})(I - KB) = I - KC^{-1}T_1,$$

and note that $KC^{-1}T_1 \in KS(E)$ by Lemma 3.23, and thus

$$AC^{-1}(I - KB)(I + KC^{-1}T_1) = I - (KC^{-1}T_1)^2, \quad (4.4)$$

with $(KC^{-1}T_1)^2 \in K(E)$ by Lemma 3.23. Similarly,

$$(I - BK)C^{-1}A = (I - BK)(I + C^{-1}K) = I - T_2C^{-1}K \quad (4.5)$$

with $T_2C^{-1}K \in K(E)$ by Lemma 3.23. We have constructed right and left regularisers for A , so A is Fredholm. ■

The following is a corollary of the above result and Lemma 3.16.

Corollary 4.13 *If $A \in L(E)$ is invertible at infinity and $A = I + K$, with $K \in S(E)$ and $P_n K \in K(E)$ for every n , then A is Fredholm.*

To show that in the first and third row of our table, the implication ‘ \Leftarrow ’ cannot hold in general, we give four examples of Fredholm operators which are not invertible (although weakly invertible) at infinity.

For this purpose we can reuse Example 4.6. If $\dim X < \infty$, then all four operators, a), b), c) and d) are Fredholm (since K is compact) and weakly invertible at infinity – but not invertible at infinity. For $\dim X = \infty$, examples a) and c) are no longer Fredholm, but b) and d) still have this property as K is still compact then.

For an operator which is Fredholm but not even weakly invertible at infinity, we go to $p = \infty$. We draw some inspiration from Example 3.12 c).

Example 4.14 Put $E = E^\infty$ and fix an element $v \in E \setminus E_0$. By the Hahn-Banach theorem there exists a bounded linear functional f on E which vanishes on E_0 and takes $f(v) = 1$. Then define operators K and A on E by

$$K : u \mapsto f(u) \cdot v \quad \text{and} \quad A := I - K.$$

Clearly, K is compact. Consequently, A is Fredholm with index zero. A simple computation shows that the kernel of A exactly consists of the multiples of v , whence $\alpha(A) = 1$. Then also $\beta(A) = 1$ follows. But how can we identify this one-dimensional cokernel?

The fact that

$$f(Au) = f(u - f(u)v) = f(u) - f(u)f(v) = 0 \quad (4.6)$$

for every $u \in E$, is a strong indication that A is not surjective (and hence, not right-invertible) at infinity. Roughly speaking, the cokernel of A must be located ‘somewhere at infinity’.

Indeed, suppose that A were weakly (right-)invertible at infinity, i.e. there exist $B \in L(E)$ and $m \in \mathbb{N}$ such that $Q_m AB = Q_m$ holds. If we rewrite this equality as $AB - P_m AB = I - P_m$, let both sides act on v and apply the functional f , we arrive at the contradiction

$$0 - 0 = f(ABv) - f(P_m ABv) = f(v) - f(P_m v) = 1 - 0$$

by (4.6), by $\text{im } P_m \subset E_0$ and by the choice of f . \square

Moreover, if $\dim X = \infty$, then Q_m is an example of an operator which is invertible at infinity but not Fredholm, where $m \in \mathbb{N}$ is arbitrary. Also note that a multiplication operator $f \mapsto bf$ on $L^p(\mathbb{R}^N)$ is invertible at infinity if and only if its symbol $b \in L^\infty(\mathbb{R}^N)$ is essentially bounded away from zero in a neighborhood of infinity – while the same operator is Fredholm if and only if this is true on the whole \mathbb{R}^N . This shows that in the right column of our table, the implication ‘ \Rightarrow ’ cannot hold in general. This example finishes the completeness discussion for our table in Figure 4.1.

Figure 4.1 also nicely illustrates another fact. In the so-called “perfect case” (see [143]), that is when $p \in \{0\} \cup (1, \infty)$ and $\dim X < \infty$, where much of this theory grew up (for instance, see [140]), everything is rather nice:

$$\left(K(E, \mathcal{P}) ; L(E, \mathcal{P}) ; \text{invertibility at } \infty \right) = \left(K(E) ; L(E) ; \text{Fredholmness} \right)$$

In fact, no one of the left hand side items even appears in [140]. The bifurcation of the theory starts when we leave that case, and it culminates when $p = 1$ or $p = \infty$ and $\dim X = \infty$. We will see below that the for us more appropriate way to follow at this bifurcation point is the left one, and that the above equality is just a coincidence that happens in the “perfect case”.

4.2 Collectively Compact Operator Theory

4.2.1 Collective Compactness: A Short Intro

The concept of collectively compact operators was introduced by Anselone and co-workers (see [4] and the references therein). A family \mathcal{K} of linear operators on a Banach space Z is called *collectively compact* if, for any sequences $(K_m) \subset \mathcal{K}$ and $(z_m) \subset Z$ with $\|z_m\| \leq 1$, there is a subsequence of $(K_m z_m)$ that converges in the norm of Z . It is immediate that every collectively compact family \mathcal{K} is bounded and that all of its members are compact operators.

There are some important features of collectively compact sets of operators. First, recall that if K is a compact operator on Z and a sequence A_m of operators on Z converges strongly to 0, then $A_m K$ converges to 0 in the operator norm on Z . But under the same assumption, even $A_m K_m$ converges to 0 in the norm for any sequence $(K_m) \subset \mathcal{K}$ provided \mathcal{K} is collectively compact. This fact was probably the motivation for the introduction of this notion. It was used by Anselone for the convergence analysis of approximation methods like the Nyström method for second kind integral equations.

Another important feature [4, Theorem 1.6] is that if $\{K_m\}_{m=1}^\infty$ is collectively compact and strongly convergent to K , then also K is compact, and the following holds:

$$I - K \text{ is invertible} \iff I - K_m \text{ is invertible for large } m, \text{ say } m > m_0, \text{ and } \sup_{m > m_0} \|(I - K_m)^{-1}\| < \infty$$

Since K and K_m are compact operators, the above is equivalent to the following statement

$$I - K \text{ is injective} \iff \exists m_0 : \inf_{m > m_0} \nu(I - K_m) > 0 \quad (4.7)$$

where

$$\nu(A) := \inf\{\|Az\| : z \in Z, \|z\| = 1\} \quad (4.8)$$

is the so-called *lower norm* of an operator A .

There are many important examples where an operator K is not compact in the norm topology but does have compactness properties in a weaker topology. To be precise, K , while not compact (mapping a neighbourhood of zero to a relatively compact set) has the property that, in the weaker topology, it maps bounded sets to sets that are relatively compact – which is exactly what we call a Montel operator. In particular, this is generically the case when K is an integral

operator on an unbounded domain with a continuous or weakly singular kernel; these properties of the kernel make K a ‘smoothing’ operator, so that K has local compactness properties, but K fails to be compact because the domain is not compact.

Anselone and Sloan [6] were the first to extend the arguments of collectively compact operator theory to tackle a case of this type, namely to study the finite section method for classical Wiener-Hopf operators on the half-axis. The arguments introduced were developed into a methodology for establishing existence from uniqueness for classes of second kind integral equations on unbounded domains and for analysing the convergence and stability of approximation methods in a series of papers by Chandler-Wilde and collaborators [29, 129, 43, 47, 116, 40, 46, 9, 10]. A particular motivation for this was the analysis of integral equation methods for problems of scattering of acoustic, elastic and electromagnetic waves by unbounded surfaces [30, 44, 185, 42, 45, 116, 40, 186, 125, 10, 37]. Other applications included the study of multidimensional Wiener-Hopf operators and, related to the Schrödinger operator, a study of Lippmann-Schwinger integral equations [43]. Related developments of the ideas of Anselone and Sloan [6] to the analysis of nonlinear integral equations on unbounded domains are described in [1, 5, 127].

In [46] Chandler-Wilde and Zhang put these ideas into the setting of an abstract Banach space Z , in which a key role is played by the notion of a generalised collectively compact family \mathcal{K} . Now the sequence $(K_m z_m)$ has a subsequence that converges in a topology that is weaker than the norm topology on Z , whenever $(K_m) \subset \mathcal{K}$ and $(z_m) \subset Y$ with $\|z_m\| \leq 1$. This notion only requires the elements of \mathcal{K} to be Montel, but not necessarily compact, operators. But still, the following similar result to (4.7) was established in [46]: If \mathcal{K} is generalised collectively compact and some additional assumptions hold (see Theorem 4.20 below), then

$$I - K \text{ is injective for all } K \in \mathcal{K} \quad \Longleftrightarrow \quad \inf_{K \in \mathcal{K}} \nu(I - K) > 0. \quad (4.9)$$

If the family \mathcal{K} satisfies some further constraints (see Theorem 4.20.b below for details), then also invertibility of $I - K$ for every $K \in \mathcal{K}$ follows from injectivity of all $K \in \mathcal{K}$.

4.2.2 The Chandler-Wilde/Zhang Approach in our Case

We now give some details of the general Banach space approach from [46], but already in a form that is slightly adapted to the setting of our sequence spaces $E = E^p(X)$ and the strict topology that was introduced earlier.

Definition 4.15 [46] We say that a set \mathcal{K} of linear operators on E is uniformly Montel (or collectively sequentially compact) on (E, s) if, for every bounded set $B \subset E$, the set $\cup_{K \in \mathcal{K}} K(B)$ is relatively compact in (E, s) .

Remark 4.16 Note that, by Lemma 2.10, $\cup_{K \in \mathcal{K}} K(B)$ is relatively compact in the strict topology iff $\cup_{K \in \mathcal{K}} K(B)$ is relatively sequentially compact in the strict topology, i.e. iff, for every sequence $(K_n) \subset \mathcal{K}$ and $(u_n) \subset B$, $(K_n u_n)$ has a strictly convergent subsequence. \square

Corollary 4.17 If $p \neq 0$ and $P_n \in K(E)$ for every n then a set \mathcal{K} of linear operators on E is uniformly Montel on (E, s) iff \mathcal{K} is bounded in $L(E)$.

Proof. This follows immediately from Corollary 3.19. \blacksquare

Following [46, Section 4], we let $iso(E)$ denote the set of isometric isomorphisms on E and call a set $S \subset iso(E)$ sufficient if, for some $n \in \mathbb{N}$ it holds that, for every $u \in E$ there exists $V \in S$ such that $2|Vu|_n \geq \|u\|$. The following example illustrates this definition:

Example 4.18 Let $E = L^p(\mathbb{R}^N) \cong \ell^p(\mathbb{Z}^N, L^p([0, 1]^N))$ with $p \in [1, \infty]$ and look at the sets $S_1 = \{V_k : k \in \mathbb{Z}^N\}$ with V_k the translation operator

$$(V_k f)(x) = f(x - k), \quad x \in \mathbb{R}^N,$$

and $S_2 = \{\Psi_k : k \in \mathbb{N}\}$, where

$$(\Psi_k f)(x) = k^{N/p} f(kx), \quad x \in \mathbb{R}^N$$

with $N/\infty := 0$. Both sets are contained in $iso(E)$. If $p = \infty$ then both S_1 and S_2 are sufficient (with $n = 1$ for instance). If $p < \infty$ then neither S_1 nor S_2 is sufficient. Although it is true that for every $u \in E$ there are $n \in \mathbb{N}$ and $V_k \in S_1$ such that $2|V_k u|_n \geq \|u\|$, there is no universal $n \in \mathbb{N}$ which is large enough to guarantee this property for all $u \in E$. \square

We say that an operator $A \in L(E)$ is *bounded below* if $\nu(A) > 0$, where $\nu(A)$ denotes the lower norm of A as defined in (4.8). $A \in L(E)$ is bounded below iff A is injective and has a closed range. Indeed, necessity is easy to see and sufficiency follows from Banach's theorem on the inverse operator saying that $A^{-1} : A(E) \rightarrow E$ acts boundedly on the range of A if that is closed. Another elementary result on the lower norm is that it depends continuously on the operator; in particular, we have

$$|\nu(A) - \nu(B)| \leq \|A - B\| \tag{4.10}$$

for all $A, B \in L(E)$.

If A is invertible then A is bounded below and $\nu(A) = 1/\|A^{-1}\|$. We will say that a set $\mathcal{A} \subset L(E)$ is *uniformly bounded below* if every $A \in \mathcal{A}$ is bounded below and if there is a $\nu > 0$ such that $\nu(A) \geq \nu$ for all $A \in \mathcal{A}$, that is

$$\|Au\| \geq \nu\|u\|, \quad A \in \mathcal{A}, u \in E.$$

For a set $\mathcal{K} \subset L(E)$, we abbreviate the set $\{I - K : K \in \mathcal{K}\}$ by $I - \mathcal{K}$.

In the following theorem we use the notation \mathcal{K}^\times to denote the set of all subsequences of sequences $(K_1, K_2, \dots) \in \mathcal{K}_1 \times \mathcal{K}_2 \times \dots$ for a fixed family of sets $\mathcal{K}_1, \mathcal{K}_2, \dots \subset L(E)$. This theorem is a slight strengthening of Theorems 4.1 and 4.4 in [46] (in [46] the condition (4.11) has ‘ $I - K_n$ bounded below’ replaced by the weaker ‘ $I - K_n$ injective’), but an examination of the proof of Theorem 4.4 in [46] shows that this slightly stronger result follows by exactly the same argument. Also note that in [39, 46] the case of a more general Banach space E with a more general approximate projection \mathcal{P} is covered.

Theorem 4.19 *Suppose that $p \neq 0$, $S \subset \text{iso}(E)$ is sufficient, $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2, \dots \subset L(E)$, and that*

- (i) $\cup_{n \geq 1} \mathcal{K}_n$ is uniformly Montel on (E, s) ;
- (ii) for every sequence $(K_n) \in \mathcal{K}^\times$, there exist a subsequence $(K_{n(m)})$ and $K \in \mathcal{K}$ such that $K_{n(m)} \xrightarrow{s} K$ as $m \rightarrow \infty$;
- (iii) for all $n \in \mathbb{N}$, it holds that $V^{-1}KV \in \mathcal{K}_n$ for all $K \in \mathcal{K}_n$ and $V \in S$;
- (iv) $I - K$ is injective for all $K \in \mathcal{K}$.

Then:

a) *There is an $n_0 \in \mathbb{N}$ such that $I - \cup_{n \geq n_0} \mathcal{K}_n$ is uniformly bounded below, i.e. there is a $\nu > 0$ such that*

$$\|(I - K)u\| \geq \nu\|u\|, \quad K \in \mathcal{K}_n, n \geq n_0, u \in E;$$

b) *If, in addition, for every $K \in \mathcal{K}$, there exists a sequence $(K_n) \in \mathcal{K}^\times$ such that $K_n \xrightarrow{s} K$ and all operators $I - K_n$ have the property*

$$I - K_n \text{ bounded below} \implies I - K_n \text{ surjective}, \quad n = 1, 2, \dots \quad (4.11)$$

then all operators in $I - \mathcal{K}$ are invertible, and

$$\sup_{K \in \mathcal{K}} \|(I - K)^{-1}\| \leq \nu^{-1}.$$

The following special case of the above theorem, obtained by setting $\mathcal{K}_1 = \mathcal{K}_2 = \dots = \mathcal{K}$ in Theorem 4.19 is worth noting. We will say that a subset $\mathcal{A} \subset \mathcal{K} \subset L(E)$ is s -dense in \mathcal{K} if, for every $K \in \mathcal{K}$, there is a sequence $(K_n) \subset \mathcal{A}$ with $K_n \xrightarrow{s} K$.

Theorem 4.20 [46, Theorem 4.5] *Suppose that $p \neq 0$, $S \subset \text{iso}(E)$ is sufficient and $\mathcal{K} \subset L(E)$ has the following properties:*

- (i) \mathcal{K} is uniformly Montel on (E, s) ;
- (ii) \mathcal{K} is s -sequentially compact;
- (iii) $V^{-1}KV \in \mathcal{K}$ for all $K \in \mathcal{K}$, $V \in S$;
- (iv) $I - K$ is injective for all $K \in \mathcal{K}$.

Then:

- a) The set $I - \mathcal{K}$ is uniformly bounded below;
- b) If in $I - \mathcal{K}$ there is an s -dense subset of surjective operators then all operators in $I - \mathcal{K}$ are surjective.

Note that in statement b), as in Theorem 4.19, all operators in $I - \mathcal{K}$ are consequently invertible, and their inverses are uniformly bounded by $1/\nu$ where $\nu > 0$ is a lower bound on all lower norms $\nu(I - K)$ with $K \in \mathcal{K}$ which exists by a).

4.3 Limit Operators

Definition 4.21 Let S^{N-1} denote the unit sphere in \mathbb{R}^N with respect to the Euclidian norm $|\cdot|_2$. If $R > 0$, $s \in S^{N-1}$ and $V \subset S^{N-1}$ is a neighbourhood of s in S^{N-1} , then

$$U_R^\infty := \{x \in \mathbb{R}^N : |x|_2 > R\} \quad (4.12)$$

is called a neighbourhood of ∞ , and

$$U_{R,V}^\infty := \{x \in \mathbb{R}^N : |x|_2 > R \text{ and } \frac{x}{|x|_2} \in V\} \quad (4.13)$$

is a neighbourhood of ∞_s . We say that a sequence $(h(m))_{m=1}^\infty \subset \mathbb{R}^N$ tends to infinity or tends to infinity in direction s , and we write $h(m) \rightarrow \infty$ or $h(m) \rightarrow \infty_s$, if, for every neighbourhood U of ∞ or ∞_s , respectively, all but finitely many elements of the sequence $(h(m))$ are in U .

In dimension $N = 1$, we will, of course, use the familiar notations $-\infty$ and $+\infty$ instead of ∞_{-1} and ∞_{+1} . Since the Euclidian norm $|\cdot|_2$ and the maximum norm $|\cdot|$ on \mathbb{R}^N are equivalent, we have that $h(m)$ tends to infinity if and only if $|h(m)| \rightarrow \infty$ as $m \rightarrow \infty$.

Recall the shift operators from Definition 3.42 and the notion of \mathcal{P} -convergence from Definition 3.44. Following [143, 106] and implementing the idea from our discussion around formula (1.1) in the introduction, we now give the following key definition.

Definition 4.22 *If $h = (h(n))_{n=1}^\infty \subset \mathbb{Z}^N$ tends to infinity and $A \in L(E)$ then*

$$A_h := \mathcal{P}\text{-}\lim_{n \rightarrow \infty} V_{-h(n)} A V_{h(n)},$$

if it exists, is called the limit operator of A with respect to the sequence h .

From what we know about \mathcal{P} -convergence it follows that the limit operator A_h for a given sequence h is unique if it exists.

The operator spectrum $\sigma^{\text{op}}(A) := \{A_h\}$ is the collection of all limit operators of A . The operator spectrum includes all limit operators A_h of A , regardless of the direction in which h tends to infinity. But sometimes this direction is of importance, and so we will split $\sigma^{\text{op}}(A)$ into many sets – the so called local operator spectra.

For every direction $s \in S^{N-1}$, the local operator spectrum $\sigma_s^{\text{op}}(A)$ is defined as the set of all limit operators A_h with $h = (h_m) \subset \mathbb{Z}^N$ and $h_m \rightarrow \infty_s$. For $n = 1$, we will abbreviate $\sigma_{-1}^{\text{op}}(A)$ and $\sigma_{+1}^{\text{op}}(A)$ by $\sigma_-^{\text{op}}(A)$ and $\sigma_+^{\text{op}}(A)$, respectively. The following result is certainly not surprising and can be found in [143, 106]. We however include the proof here because it is helpful in understanding the philosophy better.

Lemma 4.23 *For every operator $A \in L(E)$, the identity*

$$\sigma^{\text{op}}(A) = \bigcup_{s \in S^{N-1}} \sigma_s^{\text{op}}(A)$$

holds.

Proof. Clearly, every local operator spectrum is contained in the operator spectrum $\sigma^{\text{op}}(A)$. For the reverse inclusion, take some $A_h \in \sigma^{\text{op}}(A)$. The sequence $h_m/|h_m|_2$ need not converge to a point s in S^{N-1} ; but since the unit sphere S^{N-1} is compact, there is a subsequence g of h that has this property, and hence g tends to infinity in some direction $s \in S^{N-1}$. But then $A_h = A_g \in \sigma_s^{\text{op}}(A)$. ■

Example 4.24 Let $N = 1$ and $E = E^p(\mathbb{C})$ and recall the (generalised) multiplication operator $M_b \in L(E)$ with $b \in E^\infty(\mathbb{C})$. For $S \subset \mathbb{Z}$, let $\chi_S \in E^\infty(\mathbb{C})$ denote the characteristic function of S . Define $a \in E^\infty(\mathbb{C})$ by

where $[s] \leq s$ denotes the integer part of s , and set $A = M_a$. Then, where $B := \{\chi_{\{n, \dots, +\infty\}}, \chi_{\{-\infty, \dots, n\}} : n \in \mathbb{Z}\}$,

$$\sigma^{\text{op}}(A) = \{0, I, M_b : b \in B\}.$$

The figure consists of two horizontal number lines. The top number line has major tick marks at -4, -1, 0, 1, 4, 9, 16, 25, 36, 49, and 64. Above the line, there are groups of dots representing the intervals that remain at each stage of the construction. At the first stage, the middle third of the interval [-4, 64] is removed. At the second stage, the middle third of each of the two remaining intervals is removed. At the third stage, the middle third of each of the four remaining intervals is removed. Arrows point from the labels $h(1)$, $h(2)$, and $h(3)$ to the boundaries of the intervals that remain after the first, second, and third stages, respectively. The bottom number line has major tick marks at -4, -3, 0, 3, 4, and 5. Above the line, there are groups of dots representing the intervals that remain at each stage of the construction. The dots are arranged in a way that shows the iterative removal of the middle third of intervals, resulting in a fractal curve.

Figure 4.2: Functions a (top) and $b = \chi_{\{-\infty, \dots, -4\}}$ (below) from Example 4.24.

where $b := \chi_{\{-\infty, \dots, -4\}}$ (see Figure 4.2). The operator A is rich; this can be seen directly or by applying Lemma 5.1 below. \square

Example 4.25 For $E = E^p(X)$ and $b = (b(m))_{m \in \mathbb{Z}^N} \in \ell^\infty(\mathbb{Z}^N, L(X))$, look at the (generalised) multiplication operator $M_b \in L(E)$. Then, for every $k \in \mathbb{Z}^N$,

$$V_{-k}M_bV_k = M_{V_{-k}b} \quad \text{and} \quad \|M_b\| = \|b\|. \quad (4.14)$$

From these identities and (3.25), it is immediate that M_b is rich iff the set

$$\{V_kb\}_{k \in \mathbb{Z}^N} \quad (4.15)$$

is relatively sequentially compact in the strict topology on $\ell^\infty(\mathbb{Z}^N, L(X))$. It can be shown moreover [143, Theorem 2.1.16] that this is the case iff the set $\{b(m) : m \in \mathbb{Z}^N\}$ is relatively compact in $L(X)$. \square

The following theorem summarises and extends known results on the operator spectrum $\sigma^{\text{op}}(A)$ and on the relationship between A and its operator spectrum. Statements (i) and (ii) are from [140], (iii) and (iv) are from [143] and statements (v)-(vii) go back to [104, Section 3.3] and can also be found in [143, Section 1.2]. (Note that the proofs of (iii)-(vii) given in [104, 143] work for all $A \in L(E)$, although the results state a requirement for $A \in L(E, \mathcal{P})$ or make a particular choice of E , and note also that (iv) is immediate from (ii) and (iii) and that (vii) is immediate from (ii), (v) and (vi), see [38].) Thus we include only a proof of (viii) and (ix), in which $E_0 = \text{clos } E_{00} \subset E$ is as defined in Section 2.6.

For brevity, we introduce the notation

$$\mathcal{T}(A) := \{V_{-k}AV_k : k \in \mathbb{Z}^N\} \quad (4.16)$$

for the set of all translates of an operator $A \in L(E)$. Moreover, we put

$$\mathcal{V} := \{V_k : k \in \mathbb{Z}^N\}. \quad (4.17)$$

Theorem 4.26 *For every $A \in L(E)$, the following statements hold.*

- (i) *If $B \in \sigma^{\text{op}}(A)$ then $\|B\| \leq \|A\|$.*
- (ii) *If $B \in \sigma^{\text{op}}(A)$ and $k \in \mathbb{Z}^N$ then also $V_{-k}BV_k \in \sigma^{\text{op}}(A)$.*
- (iii) *$\sigma^{\text{op}}(A)$ is sequentially closed with respect to \mathcal{P} -convergence.*
- (iv) *If $B \in \sigma^{\text{op}}(A)$ then $\sigma^{\text{op}}(B) \subset \sigma^{\text{op}}(A)$.*
- (v) *A is rich iff $\mathcal{T}(A)$ is relatively \mathcal{P} -sequentially compact.*
- (vi) *If A is rich then $\sigma^{\text{op}}(A)$ is \mathcal{P} -sequentially compact.*
- (vii) *If A is rich and $B \in \sigma^{\text{op}}(A)$ then B is rich.*

(viii) If $B \in \sigma^{\text{op}}(A)$ then $\|Bx\| \geq \nu(A)\|x\|$ for $x \in E_0$, so that $\nu(B) \geq \nu(A)$ if $E = E_0$.

(ix) If $B \in \sigma^{\text{op}}(A) \cap L(E, \mathcal{P})$ is invertible then $\nu(B) \geq \nu(A)$.

Proof. (viii) If $B \in \sigma^{\text{op}}(A)$ then $B = A_h$ for some sequence $h \subset \mathbb{Z}^N$. For $m \in \mathbb{N}$ and every $u \in E$ we have that

$$\|V_{-h(n)}AV_{h(n)}P_mu\| = \|AV_{h(n)}P_mu\| \geq \nu(A)\|V_{h(n)}P_mu\| = \nu(A)\|P_mu\|.$$

Since $V_{-h(n)}AV_{h(n)} \xrightarrow{\mathcal{P}} B$, taking the limit as $n \rightarrow \infty$ we get

$$\|BP_mu\| \geq \nu(A)\|P_mu\|.$$

For $u \in E_0$ we have, by Lemma 2.6, that $P_mu \rightarrow u$ as $m \rightarrow \infty$, so the result follows.

(ix) For $m, n \in \mathbb{N}$ and $u \in E$,

$$\begin{aligned} \|P_mB^{-1}u\| &\leq \|P_mB^{-1}Q_nu\| + \|P_mB^{-1}P_nu\| \\ &\leq \|P_mB^{-1}Q_nu\| + \|B^{-1}P_nu\|. \end{aligned} \quad (4.18)$$

As $L(E, \mathcal{P})$ is inverse closed, we have that $B^{-1} \in L(E, \mathcal{P})$, so that $\|P_mB^{-1}Q_k\| \rightarrow 0$ and $\|Q_kB^{-1}P_n\| \rightarrow 0$ as $k \rightarrow \infty$, the latter implying, by Lemma 2.6, that $B^{-1}P_nu \in E_0$. Thus from (4.18) and (viii) we have that

$$\begin{aligned} \nu(A)\|P_mB^{-1}u\| &\leq \nu(A)\|P_mB^{-1}Q_nu\| + \nu(A)\|B^{-1}P_nu\| \\ &\leq \nu(A)\|P_mB^{-1}Q_nu\| + \|P_nu\|. \end{aligned}$$

Taking the limit first as $n \rightarrow \infty$ and then as $m \rightarrow \infty$, we get that $\nu(A)\|B^{-1}u\| \leq \|u\|$. We have shown that $\nu(A)\|v\| \leq \|Bv\|$, for all $v \in E$, as required. ■

Within the subspace $L(E, \mathcal{P})$ of $L(E)$, for every fixed sequence h tending to infinity, the mapping $A \mapsto A_h$ is compatible with all of addition, composition, scalar multiplication and passing to norm-limits [140]. That is, the equations

$$\begin{aligned} (A+B)_h &= A_h + B_h, & (AB)_h &= A_h B_h, \\ (\lambda A)_h &= \lambda A_h, & \left(\lim_{m \rightarrow \infty} A^{(m)} \right)_h &= \lim_{m \rightarrow \infty} A_h^{(m)} \end{aligned} \quad (4.19)$$

hold, in each case provided the limit operators on the right hand side exist.

As a consequence of (4.19), together with a diagonal argument to see the closedness, the set of rich operators $A \in L(E, \mathcal{P})$ is a Banach subalgebra of $L(E, \mathcal{P})$.

Lemma 4.27 *Let $A \in L(E)$ and B be a limit operator of A .*

- a) *If $A \in S(E)$ then $B \in S(E)$.*
- b) *If $A \in L(E, \mathcal{P})$ then $B \in L(E, \mathcal{P})$.*
- c) *If $A \in BDO(E)$ then $B \in BDO(E)$.*
- d) *If $A \in \mathcal{W}(E)$ then $B \in \mathcal{W}(E)$.*
- e) *If $A \in BO_w(E)$ for $w \in \mathbb{N}_0$ then $B \in BO_w(E)$.*

Proof. a) and b) follow from $V_k \in L(E, \mathcal{P}) \subset S(E)$ for all $k \in \mathbb{Z}^N$, from $S(E)$ and $L(E, \mathcal{P})$ being algebras and from Lemma 3.48. Finally, c), d) and e) follow from [106, Lemma 3.5 & Proposition 3.6]. ■

4.4 Collective Compactness Meets Limit Operators

We now bring the main results of the two previous sections together. Theorem 4.26 shows how nicely the operator spectrum matches the conditions made on the set \mathcal{K} in Theorem 4.20 if we put $S := \mathcal{V} = \{V_k : k \in \mathbb{Z}^N\}$:

Indeed, property (iii) of Theorem 4.20 is then guaranteed by Theorem 4.26 (ii). Moreover, if the operator under consideration is rich and in $S(E)$ then, by Theorem 4.26 (vi), its operator spectrum is \mathcal{P} -sequentially compact, and hence s -sequentially compact by Corollary 3.53 (recall that we have just seen after Theorem 4.26 that if $A \in S(E)$ then $\sigma^{\text{op}}(A) \subset S(E)$).

The only setback is that for $S = \mathcal{V}$ to be sufficient in the sense of Section 4.2, we have to restrict ourselves to the case $p = \infty$!

Bearing in mind these observations, we now apply Theorem 4.20 to the operator spectrum of an operator $A \in L(E^\infty)$. We set $K := I - A$ and apply Theorem 4.20 with

$$\mathcal{K} := \sigma^{\text{op}}(K) = I - \sigma^{\text{op}}(A) \quad \text{so that} \quad I - \mathcal{K} = \sigma^{\text{op}}(A),$$

noting that K is rich iff A is rich.

Theorem 4.28 *Suppose $E = E^\infty$, $A = I - K \in S(E)$ is rich, $\sigma^{\text{op}}(K)$ is uniformly Montel on (E, s) , and all the limit operators of A are injective. Then $\sigma^{\text{op}}(A)$ is uniformly bounded below. If, moreover, there is an s -dense subset of surjective operators in $\sigma^{\text{op}}(A)$ then all elements of $\sigma^{\text{op}}(A)$ are invertible and their inverses are uniformly bounded.*

Remark 4.29 a) Recall that, in the case $E = E^\infty$, by Corollary 3.58, the operator convergence \xrightarrow{s} is equivalent to the formally weaker notion \xrightarrow{S} of strong convergence on (E, s) , so that we can replace \xrightarrow{s} by \xrightarrow{S} and ‘ s -dense’ by ‘ S -dense’ in the above theorem and in what follows.

b) We have seen in Lemma 4.30 that $\sigma^{\text{op}}(K)$ is uniformly Montel on (E^∞, s) iff the sequence $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel. If the Banach space X is of finite dimension, then the condition that $\sigma^{\text{op}}(K)$ be uniformly Montel on (E^∞, s) is even redundant. For X finite-dimensional implies that $P_n \in K(E^\infty)$ for all n , so that $\sigma^{\text{op}}(K)$ is uniformly Montel by Corollary 4.17 and Theorem 4.26 (i). \square

We can express the condition that $\sigma^{\text{op}}(K)$ be uniformly Montel on (E, s) more directly in terms of properties of the operator K . This is the content of the next lemma, for which we introduce the following definition: call a sequence $(A_k)_{k \in \mathbb{Z}^N} \subset L(E)$ *asymptotically Montel* on (E, s) if, for every sequence $h = (h(n))_{n=1}^\infty \subset \mathbb{Z}^N$ tending to infinity and every bounded sequence $(u_n) \subset E$, it holds that $A_{h(n)}u_n$ has a strictly converging subsequence.

Lemma 4.30 *If $K \in L(E)$ and the sequence $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel on (E, s) , then $\sigma^{\text{op}}(K)$ is uniformly Montel. Conversely, if K is rich and $\sigma^{\text{op}}(K)$ is uniformly Montel then $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel.*

Proof. Suppose $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel and pick any sequence $(K_n)_{n \in \mathbb{N}} \subset \sigma^{\text{op}}(K)$. Then, by definition of the operator spectrum and \mathcal{P} -convergence, for every $n \in \mathbb{N}$ we can find $h(n) \in \mathbb{Z}^N$ with $|h(n)| \geq n$ and

$$\|P_k(K_n - V_{-h(n)}K_nV_{h(n)})\| \leq \frac{1}{n}, \quad 1 \leq k \leq n. \quad (4.20)$$

Now choose any $(u_n) \subset E$ with $\|u_n\| \leq 1$. Then, as $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel and h tends to infinity, $V_{-h(n)}K_nV_{h(n)}u_n$ has a strictly convergent subsequence. On the other hand, by (4.20),

$$P_k(K_n - V_{-h(n)}K_nV_{h(n)})u_n \rightarrow 0, \quad n \rightarrow \infty,$$

for each $k \in \mathbb{N}$, so that $(K_n - V_{-h(n)}K_nV_{h(n)})u_n \xrightarrow{s} 0$ by (2.6). Thus K_nu_n has a strictly convergent subsequence, so that, by Remark 4.16, $\sigma^{\text{op}}(K)$ is uniformly Montel.

Conversely, suppose that K is rich and $\sigma^{\text{op}}(K)$ is uniformly Montel. Take an arbitrary sequence $h = (h(n))_{n=1}^\infty \subset \mathbb{Z}^N$ which tends to infinity and an arbitrary bounded sequence $(u_n) \subset E$. Since K is rich, $(h(n))$ and (u_n) have subsequences, denoted again by $(h(n))$ and (u_n) , such that $V_{-h(n)}KV_{h(n)} \xrightarrow{\mathcal{P}} K_h \in \sigma^{\text{op}}(K)$. Thus

$$P_k(V_{-h(n)}KV_{h(n)} - K_h)u_n \rightarrow 0, \quad n \rightarrow \infty,$$

for each $k \in \mathbb{N}$, so that $(V_{-h(n)}KV_{h(n)} - K_h)u_n \xrightarrow{s} 0$. On the other hand, $K_h u_n$ has a strictly convergent subsequence since K_h is Montel. Thus $V_{-h(n)}KV_{h(n)}u_n$ has a strictly convergent subsequence. ■

Remark 4.31 Note that, clearly, $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel iff $(V_{-k}K)_{k \in \mathbb{Z}^N}$ is asymptotically Montel since $\mathcal{V} = \{V_k : k \in \mathbb{Z}^N\} \subset \text{iso}(E)$. □

An extension of Theorem 4.28 can be derived by applying Theorem 4.20 to

$$\mathcal{K} := \sigma^{\text{op}}(K) \cup \mathcal{T}(K),$$

with $\mathcal{T}(K)$ defined by (4.16), so that $I - \mathcal{K} = \sigma^{\text{op}}(A) \cup \mathcal{T}(A)$. Properties (ii) and (iii) of Theorem 4.20 can be checked in a similar way as before. Property (i) of Theorem 4.20, that \mathcal{K} is uniformly Montel on (E, s) , is equivalently characterised by any of the properties (i)-(iii) of Lemma 5.2 below, which are equivalent even for arbitrary $K \in L(E)$. Note that, for a rich operator K , by Lemma 4.30, any of these properties is moreover equivalent to $\sigma^{\text{op}}(K)$ being uniformly Montel on (E, s) and $K \in M(E)$. Then we get the following slightly enhanced version of the first part of Theorem 4.28, which in addition allows to conclude from A being injective to the closedness of the range of A .

Theorem 4.32 *Suppose $E = E^\infty$, $A = I - K \in S(E)$ is rich, K is subject to any of (i)-(iii) of Lemma 5.2, and A as well as all its limit operators are injective. Then A is bounded below and $\sigma^{\text{op}}(A)$ is uniformly bounded below.*

Note that Theorems 4.28 and 4.32 are applications of Theorem 4.20 which was just a special case of Theorem 4.19. We will now apply Theorem 4.19 directly.

Theorem 4.33 *Suppose that $E = E^\infty$, $A = I - K \in L(E)$, $A_n = I - K_n \in L(E)$ for $n \in \mathbb{N}$ and that:*

- (a) $A_n \xrightarrow{S} A$;
- (b) A_n bounded below $\Rightarrow A_n$ surjective, for each $n \in \mathbb{N}$;
- (c) $\cup_{n \in \mathbb{N}} \mathcal{T}(K_n) = \{V_{-k}K_nV_k : k \in \mathbb{Z}^N, n \in \mathbb{N}\}$ is uniformly Montel on (E, s) ;
- (d) there exists a set $\mathcal{B} \subset L(E)$, such that, for every sequence $(k(m)) \subset \mathbb{Z}^N$ and increasing sequence $(n(m)) \subset \mathbb{N}$, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and $B \in \mathcal{B}$ such that

$$V_{-k(m)}A_{n(m)}V_{k(m)} \xrightarrow{S} B \in \mathcal{B} \quad \text{as } m \rightarrow \infty;$$

(e) every $B \in \mathcal{B}$ is injective.

Then A is invertible and, for some $n_0 \in \mathbb{N}$, A_n is invertible for all $n \geq n_0$, and

$$\|A^{-1}\| \leq \sup_{n \geq n_0} \|A_n^{-1}\| < \infty. \quad (4.21)$$

Proof. Let $\mathcal{K}_n := \mathcal{T}(K_n)$, $n \in \mathbb{N}$, and set $\mathcal{K} := I - \mathcal{B}$, $S := \mathcal{V}$. Then (c) – (e) imply that conditions (i)–(iv) of Theorem 4.19 are satisfied, and (a) and (b) imply that the condition in Theorem 4.19 b) is satisfied. Thus, applying Theorem 4.19, the result follows. ■

Here and especially later, when we talk about numerical analysis, we call a sequence $(A_n) \subset L(E)$ *stable* if there is a $n_0 \in \mathbb{N}$ such that A_n is invertible for all $n \geq n_0$, and

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

Remark 4.34 Note that condition (a) in the theorem implies that $A \in S(E)$ by Lemma 3.50. Moreover, from condition (d) with $k(m) = 0$ for all $m \in \mathbb{N}$ and condition (a) again we get that $A \in \mathcal{B}$. Since $A \in S(E)$ it holds that $\sigma^{\text{op}}(A) \subset S(E)$ (see discussion at the end of Section 4.3); if also $A_n \xrightarrow{\mathcal{P}} A$ (as in Corollary 5.5) then condition (d) also implies that $\sigma^{\text{op}}(A) \subset \mathcal{B}$. To see this last claim, suppose that $\tilde{A} \in \sigma^{\text{op}}(A)$. Then there exists $(k(m)) \subset \mathbb{Z}^N$ such that $V_{-k(m)}AV_{k(m)} \xrightarrow{\mathcal{P}} \tilde{A}$ which implies, in particular, that $\|P_j(\tilde{A} - V_{-k(m)}AV_{k(m)})\| \rightarrow 0$ as $m \rightarrow \infty$, for every j . Choose the sequence $(l(m)) \subset \mathbb{N}$ such that $P_nV_{-k(m)}P_{l(m)} = P_nV_{-k(m)}$, for $n = 1, \dots, m$. Then

$$\|P_j(\tilde{A} - V_{-k(m)}AV_{k(m)})\| = \|P_j(\tilde{A} - V_{-k(m)}P_{l(m)}AV_{k(m)})\|$$

for all j and all $m > j$. Since $A_n \xrightarrow{\mathcal{P}} A$, for every m we can choose $n(m)$ such that $\|P_{l(m)}(A_{n(m)} - A)\| < m^{-1}$. Then

$$\|P_j(\tilde{A} - V_{-k(m)}A_{n(m)}V_{k(m)})\| = \|P_j(\tilde{A} - V_{-k(m)}P_{l(m)}A_{n(m)}V_{k(m)})\| \rightarrow 0$$

as $m \rightarrow \infty$, for every j , so that, by Lemma 3.52, $V_{-k(m)}A_{n(m)}V_{k(m)} \xrightarrow{s} \tilde{A}$, and so $\tilde{A} \in \mathcal{B}$. □

In the case that $A = I - K$ with $K \in S(E) \cap M(E)$, one particular choice (see e.g (7.52) below) of A_n which satisfies (a) and (b) is

$$A_n = I - KP_n.$$

For, by Lemma 3.18, $KP_n \in KS(E)$ for every n so that, by a version of Riesz Fredholm theory for TVS's (see e.g. [151]), assumption (b) holds. Further, by Corollary 3.4, for every m ,

$$\|P_m(A_n - A)\| = \|P_m K Q_n\| \rightarrow 0, \quad n \rightarrow \infty$$

so that, by Lemma 3.52, $A_n \xrightarrow{s} A$. Further, if $\mathcal{T}(K) = \{V_{-k}KV_k : k \in \mathbb{Z}^N\}$ is uniformly Montel on (E, s) , then so is $\{V_{-k}K : k \in \mathbb{Z}^N\}$ and hence also $\{V_{-k}KP_nV_k : k \in \mathbb{Z}^N, n \in \mathbb{N}\} = \cup_{n \in \mathbb{N}} \mathcal{T}(KP_n)$. Thus Theorem 4.33 has the following corollary.

Corollary 4.35 *Let $E = E^\infty$ and $A = I - K$ with $K \in S(E)$, and set $A_n = I - KP_n$ for $n \in \mathbb{N}$. Suppose that $\mathcal{T}(K)$ is uniformly Montel on (E, s) , and that there exists $\mathcal{B} \subset L(E)$ such that every $B \in \mathcal{B}$ is injective and for every sequence $(k(m)) \subset \mathbb{Z}^N$ and increasing sequence $(n(m)) \subset \mathbb{N}$, there exist subsequences, denoted again by $k(m)$ and $n(m)$, and $B \in \mathcal{B}$, such that*

$$I - V_{-k(m)}KP_{n(m)}V_{k(m)} \xrightarrow{s} B \in \mathcal{B}.$$

Then A is invertible, the sequence (A_n) is stable and (4.21) holds.

Remark 4.36 By Remark 4.34, necessarily, $A \in \mathcal{B}$ and $\sigma^{\text{op}}(A) \subset \mathcal{B}$. \square

4.5 Comments and References

The concept of invertibility at infinity, as opposed to usual Fredholmness, was first studied in [141]. In [106] it is compared with usual Fredholmness, \mathcal{P} -Fredholmness (introduced in [143]) and weak invertibility at infinity. Refinements of this study are from [39]. More recently, Seidel and Silbermann [162] gave a further characterisation of \mathcal{P} -Fredholmness in $BDO(E)$.

We have said something about the history of collectively compact operator theory in Section 4.2.1 already and therefore jump straight to limit operators. Limit operators turn up in countless situations. We will try to give a short historical account here from their first occurrences as auxiliary device until the more recent studies where they themselves are the center of attention. For a much more extensive historical review of both collectively compact operator theory and the limit operator method we refer to the introduction of [39].

The story of limit operators starts in spaces of functions on a continuous rather than discrete domain. The typical setting was originally that of a (ordinary or partial) differential operator with almost periodic coefficients. First of all, Favard

[62] showed that the condition that was subsequently named after him guarantees the existence of almost periodic solutions to a system of ODE's with almost periodic coefficients and an almost periodic right-hand side. Later, Muhamadiev [121] proved that Favard's condition implies the invertibility of Favard's almost periodic differential operator considered as operator from $BC^1(\mathbb{R}, \mathbb{R}^n)$ to $BC(\mathbb{R}, \mathbb{R}^n)$. Extensions of Muhamadiev's result to wider classes of almost periodic operators can be found in [122, 123, 169, 170, 95], for example. For operators A with almost periodic coefficients, the connection between A and its limit operators is a lot stronger than in more general settings. In particular, all limit operators of A are norm-limits of translates of A , including the operator A itself.

In [121], Muhamadiev went on to study matrix ODE's on the real line with merely bounded and uniformly continuous coefficients which lead him to define limit operators as limits of translates of the operator A with respect to what we call \mathcal{P} -convergence now. In this wider setting he states that Favard's condition implies the invertibility of all limit operators from $BC^1(\mathbb{R}, \mathbb{R}^n)$ to $BC(\mathbb{R}, \mathbb{R}^n)$. Later on, Muhamadiev [122] and Shubin [170] studied elliptic differential operators A with almost periodic coefficients. For infinitely smooth coefficients, Shubin provides a proof of Muhamadiev's result [122] that the Favard condition is equivalent to the invertibility of A on $BC^\infty(\mathbb{R}^N, \mathbb{R})$. In [123], Muhamadiev showed that, for Hölder continuous coefficients, Favard's condition is equivalent to A being Φ_+ -semi Fredholm between an appropriate pair of spaces of bounded Hölder continuous functions. Similarly and much more recently, Volpert and Volpert show that, for a general class of scalar elliptic partial differential operators A on an unbounded domain but also for systems of such, the Favard condition is equivalent to the Φ_+ -semi Fredholmness of A on appropriate Hölder [179, 180] or Sobolev [178, 180] spaces. Lange and Rabinovich [96] state a corresponding result about semi Fredholmness of band-dominated operators in the discrete scalar-valued $\ell^\infty(\mathbb{Z}^N, \mathbb{C})$ setting.

In the last 10 years, limit operators of band-dominated operators on discrete ℓ^p spaces with values in a complex Banach space X and $p \in (1, \infty)$ have been extensively studied by Rabinovich, Roch and Silbermann [140, 143]. The author [104, 106] then extended some of their results to $p \in \{1, \infty\}$. The reformulation of the property of an operator A to be rich in terms of sequential \mathcal{P} -compactness of the operator spectrum $\sigma^{\text{op}}(A)$ of A in [104] then was the starting point for an improvement of the limit operator results by combining them with the generalised collectively compact operator theory of Chandler-Wilde and Zhang [46]. The results of this approach, so far, are in [38, 39].

For the C^* -algebra case $E = \ell^2(G, \mathbb{C})$ with a locally compact abelian or a finitely generated discrete (possibly non-abelian) group G , the limit operator concept is nicely presented in [70, 155]. Limit operators A_h of A are here intro-

duced via extension of the map $g \mapsto V_h A V_h^*$ from $h \in G$ to h in the Stone-Čech boundary ∂G of G , where $(V_h u)(g) = u(hg)$ is the corresponding shift operator (see Section 3.5.2 of [106] for a quick introduction to this point of view).

As mentioned above, there is nowadays a vast amount of literature where limit-operator-type ideas play an essential role. Here are some examples from the bulk of literature on spectral properties of Schrödinger and Jacobi operators: [128, 51, 3, 114, 67, 68, 69, 70, 137, 100, 99, 149, 150]. Much of this work is along the lines of formula (1.2) (often with the closure taken on the right-hand side); the three last papers also shed some light on the role of limit operators in the study of absolutely continuous spectrum.

Chapter 5

Fredholm Theory of Band-Dominated Operators

In this chapter we focus on the case of band-dominated operators A on a space $E = E^p(X)$, where we give criteria for invertibility at infinity and Fredholmness of A in terms of its limit operators. From Proposition 4.9 we know that for band-dominated operators the notions of invertibility at infinity and \mathcal{P} -Fredholmness coincide.

5.1 More Preliminaries

5.1.1 Rich Band-Dominated Operators

Recall that an operator $A \in L(E)$ is a band operator if one of the following equivalent conditions holds,

- A is induced by an infinite band matrix M ;
- A is a finite sum-product of generalised multiplication operators and shifts;
- A can be written in the form

$$A = \sum_{|k| \leq w} M_{b_k} V_k, \quad (5.1)$$

where $b_k \in \ell^\infty(\mathbb{Z}^N, L(X))$ with $|k| \leq w$ is the k th diagonal of M ,

and that A is a band-dominated operator if it is the norm limit of a sequence of band operators.

As a consequence of (4.19), an operator $A \in BO(E)$, which has the form (5.1), is a rich operator iff the multiplication operators M_{b_k} are rich for all $|k| \leq w$, i.e. iff (see Example 4.25) the set $\{b_k(m) : m \in \mathbb{Z}^N\}$ is relatively compact in $L(X)$ for every k . Further, if $A \in BDO(E)$, in which case $A_n \rightrightarrows A$ for some $(A_n) \subset BO(E)$, A is rich if each A_n is rich. An immediate consequence is the following lemma (which is [143, Corollary 2.1.17]).

Lemma 5.1 *If X is finite-dimensional then every band-dominated operator is rich.*

5.1.2 When is $\mathcal{T}(K)$ Uniformly Montel?

In the results of Section 4.4 we have conditions such as ‘if $\mathcal{T}(K)$ is uniformly Montel on (E, s) ’ or ‘if $\sigma^{\text{op}}(K)$ is uniformly Montel on (E, s) ’, where $\mathcal{T}(K)$ is defined by (4.16). For band-dominated operators K , we have the following result; in fact the equivalence of (i), (ii) and (iii) in the next lemma is true even for arbitrary $K \in L(E)$.

Lemma 5.2 *If $K \in BDO(E)$, with $[K] = (\kappa_{ij})_{i,j \in \mathbb{Z}^N}$ the matrix representation of K , then the following statements are equivalent.*

- (i) $\{V_{-k}KV_k : k \in \mathbb{Z}^N\} \cup \sigma^{\text{op}}(K)$ is uniformly Montel on (E, s) .
- (ii) $\{V_{-k}KV_k : k \in \mathbb{Z}^N\}$ is uniformly Montel on (E, s) .
- (iii) $(V_{-k}KV_k)_{k \in \mathbb{Z}^N}$ is asymptotically Montel on (E, s) and $K \in M(E)$.
- (iv) The set $\{\kappa_{ij} : i, j \in \mathbb{Z}^N\} \subset L(X)$ is collectively compact.
- (v) The set $\{\kappa_{ij} : i, j \in \mathbb{Z}^N, i - j = d\} \subset L(X)$ is collectively compact for every $d \in \mathbb{Z}^N$.

If X is finite-dimensional, then (i)–(v) are also equivalent to:

- (vi) The set $\{\kappa_{ij} : i, j \in \mathbb{Z}^N, i - j = d\} \subset L(X)$ is bounded for every $d \in \mathbb{Z}^N$.

Proof. It is clear from the definitions that (i) \Rightarrow (ii), (ii) \Rightarrow (iii) and that (iv) \Rightarrow (v). By Lemma 4.30, (ii) implies (i).

Suppose now that (iii) holds and that $h = (h(n))_{n=1}^\infty \subset \mathbb{Z}^N$ and that $(u_n) \subset E$ is bounded. If h does not have a subsequence that tends to infinity, then h is bounded, and hence it has a subsequence that is constant. In the case that h has

a subsequence that tends to infinity, $(V_{-h(n)}KV_{h(n)}u_n)$ has a strictly convergent subsequence since $(V_{-k}KV_k)$ is asymptotically Montel. In the case that h has a constant subsequence, $(V_{-h(n)}KV_{h(n)}u_n)$ has a strictly convergent subsequence since $K \in M(E)$. In either case, we have shown that (ii) holds.

Next suppose that (ii) holds and that $i = (i(n))_{n=1}^\infty \subset \mathbb{Z}^N$, $j = (j(n))_{n=1}^\infty \subset \mathbb{Z}^N$, and that $(x_n) \subset X$ is bounded. For $n \in \mathbb{Z}^N$ define $(u_n) \in E$ by setting the $i(n) - j(n)$ entry of u_n equal to x_n and setting the other entries to zero. Then (u_n) is bounded and the zeroth entry of $(V_{-i(n)}KV_{i(n)}u_n)$ is $\kappa_{i(n),j(n)}x_n$. Since $\{V_{-k}KV_k : k \in \mathbb{Z}^N\}$ is uniformly Montel, $(V_{-i(n)}KV_{i(n)}u_n)(0) = \kappa_{i(n),j(n)}x_n$ has a convergent subsequence. Since i , j , and (x_n) were arbitrary sequences, we have shown that (iv) holds.

Finally, suppose that (v) holds. Then the set $\{\kappa_{ij} : i, j \in \mathbb{Z}^N, |i - j| \leq w\}$ is collectively compact for every $w \in \mathbb{N}$. For every $M \in \mathbb{N}$, every $h = (h(n))_{n=1}^\infty \subset \mathbb{Z}^N$, and every bounded sequence $(u_n) \subset E$, we have that the i th component of $(V_{-h(n)}KV_{h(n)}P_M u_n)$ is

$$\sum_{|j| \leq M} \kappa_{i+h(n),j+h(n)} u_n(j).$$

Since $\{\kappa_{ij} : i, j \in \mathbb{Z}^N, |i - j| \leq w\}$ is collectively compact for each w , it follows that the i th component of $(V_{-h(n)}KV_{h(n)}P_M u_n)$ has a convergent subsequence for every $M \in \mathbb{Z}$. Thus, by a diagonal argument, $(V_{-h(n)}KV_{h(n)}P_M u_n)$ has a strictly convergent subsequence, for every $M \in \mathbb{N}$. Again by a diagonal argument, we can find subsequences of h and (u_n) , which we will denote again by h and (u_n) , such that $(V_{-h(n)}KV_{h(n)}P_n u_n)$ converges strictly to some $u \in E$, so that $P_m V_{-h(n)}KV_{h(n)}P_n u_n \rightarrow P_m u$ as $n \rightarrow \infty$, for each m . Now [143], since K is band-dominated, it holds for every $m \in \mathbb{N}$ that $P_m V_{-k}KV_k Q_n \rightrightarrows 0$ as $n \rightarrow \infty$, uniformly in $k \in \mathbb{Z}^N$. Thus $P_m V_{-h(n)}KV_{h(n)}u_n \rightarrow P_m u$, for every $m \in \mathbb{N}$, so that $V_{-h(n)}KV_{h(n)}u_n \xrightarrow{s} u$. We have shown that (ii) holds.

The equivalence of (vi) and (v) under the condition $\dim X < \infty$ is obvious since, in that case, a subset of X is relatively compact iff it is bounded. ■

For brevity, and because we will frequently refer to this class of operators in what follows, let us denote the set of all operators $K \in BDO(E)$ which are subject to the (equivalent) properties (i)–(v) of Lemma 5.2 by $UM(E)$.

Moreover, let us write $\mathcal{M}(K)$ for the set

$$\mathcal{M}(K) := \{\kappa_{ij} : i, j \in \mathbb{Z}^N\} \subset L(X) \quad (5.2)$$

of all matrix entries of an operator $K \in BDO(E)$ with $[K] = (\kappa_{ij})$. By Lemma 5.2 we have that $K \in UM(E)$ iff $K \in BDO(E)$ and $\mathcal{M}(K)$ is collectively compact in $L(X)$.

Lemma 5.3 *If $A \in UM(E)$ and B is a limit operator of A then $B \in UM(E)$.*

Proof. By the definition of a limit operator, the set $\mathcal{M}(B)$ is contained in the closure of $\mathcal{M}(A)$. Consequently, $\mathcal{M}(B)$ is collectively compact if $\mathcal{M}(A)$ is collectively compact. Moreover, by Lemma 4.27, $B \in BDO(E)$ if $A \in BDO(E)$. ■

Lemma 5.4 *The following statements hold.*

- (a) *The set $UM(E)$ is a Banach subspace of $BDO(E) \cap M(E)$.*
- (b) *In particular, $UM(E) = BDO(E)$ if X is finite-dimensional.*
- (c) *If $K \in UM(E)$ and $A \in BDO(E)$, then $KA \in UM(E)$.*
- (d) *If M_b is rich and $K \in UM(E)$, then $M_b K \in UM(E)$.*

Proof. (a): By its definition, we have that $UM(E) \subset BDO(E)$, and from Lemma 5.2 (iii) we get that $UM(E) \subset M(E)$. For the rest of this proof, we will use property (ii) from Lemma 5.2 to characterise the set $UM(E)$.

If $S, T \in UM(E)$ and $\lambda \in \mathbb{C}$, then clearly $\lambda S + T \in UM(E)$ since

$$\{V_{-k}(\lambda S + T)V_k : k \in \mathbb{Z}^N\} \subset \lambda\{V_{-k}SV_k : k \in \mathbb{Z}^N\} + \{V_{-k}TV_k : k \in \mathbb{Z}^N\}$$

is uniformly Montel.

If $T_1, T_2, \dots \in UM(E)$ are such that $T_n \rightrightarrows T$, then also $T \in UM(E)$. To see this, take a sequence $k = (k(1), k(2), \dots) \subset \mathbb{Z}^N$ and a sequence $(u_1, u_2, \dots) \subset E$ with $\mu := \sup \|u_\ell\| < \infty$. By a simple diagonal argument, we can pick a strictly monotonously increasing sequence $s = (s(1), s(2), \dots) \subset \mathbb{N}$ such that

$$V_{-k(s(\ell))} T_n V_{k(s(\ell))} u_{s(\ell)}$$

converges strictly as $\ell \rightarrow \infty$ for every $n \in \mathbb{N}$. Let us denote the strict limit by y_n , respectively. From

$$\|y_{n_1} - y_{n_2}\| \leq \sup_{\ell} \|V_{-k(s(\ell))}(T_{n_1} - T_{n_2})V_{k(s(\ell))}u_{s(\ell)}\| \leq \|T_{n_1} - T_{n_2}\| \cdot \mu$$

we see that (y_n) is a Cauchy sequence in E and therefore converges, to $y \in E$, say. But then $V_{-k(s(\ell))} T V_{k(s(\ell))} u_{s(\ell)} \xrightarrow{s} y$ as $\ell \rightarrow \infty$. Indeed, for all $M, n \in \mathbb{N}$,

$$\begin{aligned} & \|P_M(V_{-k(s(\ell))} T V_{k(s(\ell))} u_{s(\ell)} - y)\| \\ & \leq \|P_M(V_{-k(s(\ell))} T_n V_{k(s(\ell))} u_{s(\ell)} - y_n)\| + \|P_M(y_n - y)\| \\ & \quad + \|P_M(V_{-k(s(\ell))} (T - T_n) V_{k(s(\ell))} u_{s(\ell)})\| \\ & \leq \|P_M(V_{-k(s(\ell))} T_n V_{k(s(\ell))} u_{s(\ell)} - y_n)\| + \|y_n - y\| + \|T - T_n\| \cdot \mu \end{aligned}$$

holds. But, for every choice of $M, n \in \mathbb{N}$, the first term goes to zero as $\ell \rightarrow \infty$, and the second and third term can be made as small as desired by choosing n sufficiently large.

(b): If $K \in BDO(E)$ then property (vi) of Lemma 5.2 is automatically the case. Since this is equivalent to properties (i)–(v) of the same lemma if X is finite-dimensional, we get that $K \in UM(E)$ then.

(c), (d): Let $K \in UM(E)$, $A \in BDO(E)$ and $b \in E^\infty(L(X))$ such that M_b is rich. Take a sequence $k = (k(1), k(2), \dots) \subset \mathbb{Z}^N$ and a bounded sequence $(u_1, u_2, \dots) \subset E$. Now, for every $\ell \in \mathbb{N}$, put $y_\ell := V_{-k(\ell)} A V_{k(\ell)} u_\ell$. Since (y_ℓ) is bounded, $\{V_{-m} K V_m : m \in \mathbb{Z}^N\}$ is uniformly Montel and $\{V_{-m} b : m \in \mathbb{Z}^N\}$ is relatively sequentially compact in the strict topology on $E^\infty(L(X))$ (since M_b is rich, see Example 4.25), we can pick a strictly monotonously increasing sequence $s = (s(1), s(2), \dots) \subset \mathbb{N}$ such that both $V_{-k(s(\ell))} K V_{k(s(\ell))} y_{s(\ell)}$ and $V_{-k(s(\ell))} b$ converge strictly as $\ell \rightarrow \infty$. But then $V_{-k(s(\ell))} (M_b K A) V_{k(s(\ell))} u_{s(\ell)}$ converges strictly as $\ell \rightarrow \infty$ since, for every $m \in \mathbb{N}$,

$$\begin{aligned} & P_m V_{-k(s(\ell))} (M_b K A) V_{k(s(\ell))} u_{s(\ell)} \\ &= P_m (V_{-k(s(\ell))} M_b V_{k(s(\ell))}) (V_{-k(s(\ell))} K V_{k(s(\ell))}) (V_{-k(s(\ell))} A V_{k(s(\ell))} u_{s(\ell)}) \\ &= M_{P_m V_{-k(s(\ell))} b} P_m (V_{-k(s(\ell))} K V_{k(s(\ell))} y_{s(\ell)}) \end{aligned}$$

converges in norm as $\ell \rightarrow \infty$, i.e. $M_b K A \in UM(E)$. ■

The following is a simple corollary (recall Corollary 3.53 and note that $BDO(E) \subset S(E)$) of Theorem 4.33 and Lemma 5.2 that is often already strong enough (see e.g. [38]) for what we have in mind.

Corollary 5.5 *Let $E = E^\infty$ and $A \in BDO(E)$, and take a sequence $A_n \in BDO(E)$, $n \in \mathbb{N}$, with*

- (a) $A_n \xrightarrow{\mathcal{P}} A$;
- (b) A_n injective $\Rightarrow A_n$ surjective, for each $n \in \mathbb{N}$;
- (c) $\cup_{n \in \mathbb{N}} \mathcal{M}(A_n - I)$ is collectively compact in $L(X)$;
- (d) there exists a set $\mathcal{B} \subset L(E)$, such that, for every sequence $(k(m)) \subset \mathbb{Z}^N$ and increasing sequence $(n(m)) \subset \mathbb{N}$, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and $B \in \mathcal{B}$ such that

$$V_{-k(m)} A_{n(m)} V_{k(m)} \xrightarrow{\mathcal{P}} B \in \mathcal{B} \quad \text{as } m \rightarrow \infty;$$

- (e) every $B \in \mathcal{B}$ is injective.

Then A is invertible, the sequence (A_n) is stable and (4.21) holds.

5.1.3 The Operator Spectra of A^* and $A|_{E_0}$

The purpose of the following two lemmas is to prove that, for every $A \in L(E^1(X))$, the operator spectra $\sigma^{\text{op}}(A) \subset L(E^1(X))$ and $\sigma^{\text{op}}(A^*) \subset L(E^\infty(X^*))$ correspond elementwise in terms of adjoints.

Lemma 5.6 *If $A \in L(E^1(X))$, then*

$$\sigma^{\text{op}}(A^*) = \{B^* : B \in \sigma^{\text{op}}(A)\}.$$

Proof. It is a standard result that $B = A_h \in \sigma^{\text{op}}(A)$ implies $B^* = (A_h)^* = (A^*)_h \in \sigma^{\text{op}}(A^*)$ (see, e.g. [106, Proposition 3.4 e]).

For the reverse implication, suppose $C \in \sigma^{\text{op}}(A^*) \subset L(E^\infty(X^*))$. Then

$$(V_{-h(m)}AV_{h(m)})^* = V_{-h(m)}A^*V_{h(m)} \xrightarrow{\mathcal{P}} C$$

as $m \rightarrow \infty$ for some sequence $h(1), h(2), \dots \rightarrow \infty$ in \mathbb{Z}^N . We will show in Lemma 5.7 that then $C = B^*$ and $V_{-h_m}AV_{h_m} \xrightarrow{\mathcal{P}} B$, i.e. $B \in \sigma^{\text{op}}(A)$. ■

Lemma 5.7 *The set of operators in $L(E^\infty(X^*))$ that possess a preadjoint in $L(E^1(X))$ is sequentially closed under \mathcal{P} -convergence; that is, if $A_1, A_2, \dots \in L(E^1(X))$ and $A_m^* \xrightarrow{\mathcal{P}} C$ on $E^\infty(X^*)$, then there is a $B \in L(E^1(X))$ such that $C = B^*$; moreover, $A_m \xrightarrow{\mathcal{P}} B$ on $E^1(X)$.*

Proof. From $A_m^* \xrightarrow{\mathcal{P}} C$ in $L(E^\infty(X^*))$ and Lemma 3.45 we get that there is an $M > 0$ such that

$$\|A_m\| = \|A_m^*\| \leq M, \quad m \in \mathbb{N}. \quad (5.3)$$

Moreover, for every $k \in \mathbb{N}$, it holds that

$$P_k(A_m^* - C) \rightrightarrows 0 \quad \text{as} \quad m \rightarrow \infty. \quad (5.4)$$

So we get that $(P_k A_m^*)_{m=1}^\infty$ is a Cauchy sequence in $L(E^\infty(X^*))$ and therefore $(A_m P_k)_{m=1}^\infty$ is one in $L(E^1(X))$, for every fixed $k \in \mathbb{N}$. Denote the norm-limit of the latter sequence by $B_k \in L(E^1(X))$. As a consequence of (5.3) we get that

$$\|B_k\| = \left\| \lim_{m \rightarrow \infty} A_m P_k \right\| \leq \sup_m \|A_m P_k\| \leq M, \quad k \in \mathbb{N}. \quad (5.5)$$

From $A_m P_k \rightrightarrows B_k$ we get that $B_k P_k = B_k$ and, even more than this, that

$$B_r P_k = \lim_{m \rightarrow \infty} A_m P_r P_k = \lim_{m \rightarrow \infty} A_m P_k = B_k, \quad r \geq k. \quad (5.6)$$

We will now show that the sequence B_1, B_2, \dots strongly converges in $E^1(X)$. Therefore, take an arbitrary $u \in E^1(X)$ and let us verify that $(B_m u)$ is a Cauchy sequence in $E^1(X)$. So choose some $\varepsilon > 0$. Since $Q_m u \rightarrow 0$ on $E^1(X)$, there is an $N \in \mathbb{N}$ such that

$$\|Q_N u\| < \frac{\varepsilon}{2M}. \quad (5.7)$$

Now, for all $k, m \geq N$, the following holds

$$\begin{aligned} \|B_k u - B_m u\| &\leq \|(B_k - B_m)P_N u\| + \|(B_k - B_m)Q_N u\| \\ &\leq \|(B_k P_N - B_m P_N)u\| + \|B_k - B_m\| \cdot \|Q_N u\| \\ &\leq \|(B_N - B_N)u\| + (\|B_k\| + \|B_m\|) \cdot \|Q_N u\| < \varepsilon \end{aligned}$$

by (5.6), (5.5) and (5.7). Consequently, $(B_m u)$ is a Cauchy sequence in $E^1(X)$. Let us denote its limit in $E^1(X)$ by Bu , thereby defining an operator $B \in L(E^1(X))$. Passing to the strong limit as $r \rightarrow \infty$ in (5.6), we get

$$BP_k = B_k, \quad k \in \mathbb{N}. \quad (5.8)$$

Summing up, we have $A_m P_k \rightrightarrows B_k = BP_k$, and hence $(A_m - B)P_k \rightrightarrows 0$ as $m \rightarrow \infty$, for all $k \in \mathbb{N}$. Passing to adjoints in the latter gives $P_k(A_m^* - B^*) \rightrightarrows 0$ in $L(E^\infty(X^*))$ as $m \rightarrow \infty$. If we subtract this from (5.4) we get $P_k(B^* - C) = 0$ for all $k \in \mathbb{N}$, and consequently $C = B^*$, by Lemma 1.30 a) in [106]. From $A_m^* \xrightarrow{\mathcal{P}} C = B^*$ we then conclude

$$\|(A_m - B)P_k\| = \|P_k(A_m^* - B^*)\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty$$

and

$$\|P_k(A_m - B)\| = \|(A_m^* - B^*)P_k\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty$$

for every $k \in \mathbb{N}$, which, together with (5.3) and again Lemma 3.45, proves $A_m \xrightarrow{\mathcal{P}} B$. ■

Our next statement is similar to Lemma 5.6, but with restriction from E to E_0 instead of passing to the adjoint operator.

Lemma 5.8 *If $A \in L(E, \mathcal{P})$, then the limit operators of the restriction $A_0 := A|_{E_0}$ are the restrictions of the limit operators of A ; precisely,*

$$\sigma^{\text{op}}(A_0) = \{B|_{E_0} : B \in \sigma^{\text{op}}(A)\}. \quad (5.9)$$

In particular, the invertibility of all limit operators of A_0 in E_0 with uniform boundedness of their inverses is equivalent to the same property for the limit operators of A in E .

Proof. The proof of (5.9) consists of two observations. The first one is that $(V_{-\alpha}AV_{\alpha})|_{E_0} = V_{-\alpha}A_0V_{\alpha}$ for all $\alpha \in \mathbb{Z}^N$, and the second one is that $A_m|_{E_0} \xrightarrow{\mathcal{P}} A_0$ on E_0 iff $A_m \xrightarrow{\mathcal{P}} A$ on E , for all $A_1, A_2, \dots \in L(E, \mathcal{P})$ since

$$\|P_k(A_m|_{E_0} - A_0)\| = \|(P_k(A_m - A))|_{E_0}\| = \|P_k(A_m - A)\|$$

and its symmetric counterpart hold for all $k \in \mathbb{N}$ by the norm equality in Lemma 3.15. The proof of the second sentence of the lemma now follows from (5.9), Corollary 3.40 and the norm equality in Lemma 3.15 again. ■

5.2 Main Theorems on Fredholmness and Invertibility at Infinity in $BDO(E)$

5.2.1 The General Case, $E = E^p(X)$

For a rich band-dominated operator A , the operator spectrum $\sigma^{\text{op}}(A)$ contains enough information to characterise the invertibility at infinity of A , which is the content of (iii) in the next theorem.

Theorem 5.9 a) *Let A be a rich band-dominated operator on $E = E^p(X)$ with a Banach space X and some $p \in \{0\} \cup [1, \infty]$. Then the following statements hold.*

- (i) *If A is Fredholm and $p \neq \infty$ then A is invertible at infinity;*
- (ii) *If A is invertible at infinity and either X is finite-dimensional or $A = C + K$ with $C \in BDO(E)$ invertible and $K \in M(E)$ then A is Fredholm;*
- (iii) *A is invertible at infinity if and only if all limit operators of A are invertible and their inverses are uniformly bounded;*
- (iv) *The condition on uniform boundedness in (iii) is redundant if $p \in \{0, 1, \infty\}$;*
- (v) *It holds that $\text{spec}(B) \subset \text{spec}(A)$ for all $B \in \sigma^{\text{op}}(A)$, indeed $\text{spec}(B) \subset \text{spec}_{\text{ess}}(A)$, for $p \neq \infty$;*
- (vi) *It holds that $\text{spec}_{\varepsilon}(B) \subset \text{spec}_{\varepsilon}(A)$, for all $B \in \sigma^{\text{op}}(A)$ and $\varepsilon > 0$.*

b) *In the case $p = \infty$ if, in addition, it holds that X has a predual X^{\natural} , and A has a preadjoint, $A^{\natural} \in L(E^{\natural})$, where $E^{\natural} = E^1(X^{\natural})$, then (i) and (v) also apply for $p = \infty$; that is, A being Fredholm implies A being invertible at infinity, so that $\text{spec}(B) \subset \text{spec}_{\text{ess}}(A) \subset \text{spec}(A)$ for all $B \in \sigma^{\text{op}}(A)$.*

Remark 5.10 This theorem makes several additions and simplifications to previously known results:

- (i), and therefore (v), is probably new for $p = 1$, and so is statement b).
- (ii) is a slight extension of Proposition 2.15 of [106].
- (iii) does not assume the existence of a preadjoint operator (unlike Theorem 1 in [104] and [106]) if $p = \infty$.
- (iv) is probably new for $p = 0$.
- (vi) was only known when $p = 2$ and X is a Hilbert space. For this setting, it follows from Theorem 6.3.8 (b) of [143] which, in fact, states the stronger result that the closure of the union of all $\text{spec}_\varepsilon(B)$ with $B \in \sigma^{\text{op}}(A)$ is equal to the ε -pseudospectrum of the coset of A modulo $K(E, \mathcal{P})$.

So, from (i), (ii) and b) we get that, for rich band-dominated operators, our table in Figure 4.1 can be improved as follows:

- ‘invertible at infinity \Leftarrow Fredholm’ holds if $p < \infty$ or if $p = \infty$ and A has a preadjoint. The latter holds in particular if X is reflexive (see Lemma 3.43), so that e.g. in the first column of Figure 4.1, all arrows are both ways ‘ \Leftrightarrow ’.
- ‘invertible at infinity \Rightarrow Fredholm’ holds if $\dim X < \infty$ or if $A = C + K$ with $C \in BDO(E)$ invertible and $K \in M(E)$.

Also note that our Lemma 5.6 fills a gap in the proof of [106, Proposition 3.6 a)] that is used in the proof of [106, Theorem 3.109] to deal with the case $p = 1$.

□

Proof of Theorem 5.9. b) Suppose the predual X^\diamond and preadjoint A^\diamond exist and that A is Fredholm on $E = E^\infty(X)$. By Proposition 3.37 (note that $BDO(E) \subset L(E, \mathcal{P}) \subset S(E) \cap L_0(E)$), we have that $A_0 := A|_{E_0}$ is Fredholm on $E_0 = E^0(X)$. From Lemma 3.11 (i) (also see Figure 4.1) we get that then A_0 is invertible at infinity on E_0 , which, by Proposition 4.9, means that we have $A_0 B_0 = I + S_0$ and $B_0 A_0 = I + T_0$ for some $B_0 \in L(E_0, \mathcal{P})$ and $S_0, T_0 \in K(E_0, \mathcal{P})$. If we use Lemma 3.15 to extend both sides of these two equalities to operators on E , then we get that A is invertible at infinity on E .

a) (iii) For $p \in \{0\} \cup (1, \infty)$ we refer the reader to [143, Theorem 2.2.1], and for $p = 1$ (and also $p \in (1, \infty)$) to [106, Theorem 1]. It remains to study the case $p = \infty$. The ‘if’ part of statement (iii) is Proposition 3.16 in [106] (which

does not use the existence of a preadjoint). For the ‘only-if’ part of (iii) we replace Proposition 3.12 from [106] (which needs the preadjoint) by the following argument. Suppose A is invertible at infinity on $E = E^\infty$. By Proposition 4.9, there are $B \in BDO(E) \subset L(E, \mathcal{P})$ and $S, T \in K(E, \mathcal{P})$ with $AB = I + S$ and $BA = I + T$. Restricting both sides in both equalities to E_0 we get that, by Lemma 3.15, $A_0 := A|_{E_0}$ is invertible at infinity on E_0 , which, by our result (iii) for $p = 0$, implies that all limit operators of A_0 are invertible on E_0 and their inverses are uniformly bounded. From Lemma 5.8 we now get that also all limit operators of A are invertible on E and their inverses are uniformly bounded.

Statement (iv) for $p \in \{1, \infty\}$ is [106, Theorem 3.109]. Precisely, the part for $p = \infty$ follows immediately from [106, Proposition 3.108], and the $p = 1$ part is a consequence of this and Lemma 5.6. Indeed, if all $B \in \sigma^{\text{op}}(A)$ are invertible on $E^1(X)$ then also all their adjoints $C = B^*$ are invertible on $E^\infty(X^*)$, which, by Lemma 5.6, are all elements of $\sigma^{\text{op}}(A^*)$. Since $A^* \in BDO(E^\infty(X^*))$ is rich as well, we know from the results about $p = \infty$ that

$$\begin{aligned} \sup_{B \in \sigma^{\text{op}}(A)} \|B^{-1}\| &= \sup_{B \in \sigma^{\text{op}}(A)} \|(B^{-1})^*\| = \sup_{B \in \sigma^{\text{op}}(A)} \|(B^*)^{-1}\| \\ &= \sup_{C = B^* \in \sigma^{\text{op}}(A^*)} \|C^{-1}\| < \infty \end{aligned}$$

since $B \in \sigma^{\text{op}}(A)$ iff $C = B^* \in \sigma^{\text{op}}(A^*)$, by Lemma 5.6. The statement (iv) for $p = 0$ follows immediately from Lemma 5.8 (applied to the extension of A) and the result for $p = \infty$.

(i) For $p \in \{0\} \cup (1, \infty)$ this follows immediately from Lemma 3.11 (i) (also see Figure 4.1). So let $p = 1$ and suppose A is Fredholm on $E = E^1(X)$. Then A^* is Fredholm on $E^\infty(X^*)$. From part b) of this theorem we have that A^* is invertible at infinity on $E^\infty(X^*)$. By (iii), all limit operators of A^* are invertible on $E^\infty(X^*)$. By Lemma 5.6 this implies that all limit operators of A are invertible on $E^1(X)$, which, by (iii) and (iv), shows that A is invertible at infinity.

(ii) Suppose A is invertible at infinity. If $\dim X < \infty$ then Lemma 3.11 (ii) (also see Figure 4.1) implies that A is Fredholm. Alternatively, suppose that $A = C + K$ with $C \in BDO(E)$ invertible and $K \in M(E)$. From $C \in BDO(E)$ we get, by Proposition 3.65, that $C^{-1} \in BDO(E) \subset S(E)$. Moreover, $K = A - C \in BDO(E) \subset S(E)$ implies that $K \in S(E) \cap M(E)$ so that A is subject to the constraints in Theorem 4.12 which proves that A is Fredholm.

(v) For arbitrary $\lambda \in \mathbb{C}$, $\lambda I - B \in \sigma^{\text{op}}(\lambda I - A)$ iff $B \in \sigma^{\text{op}}(A)$. So it suffices to show that Fredholmness of a rich band-dominated operator (for $p \neq \infty$) implies invertibility of its limit operators. But this is a consequence of (i) and (iii).

(vi) From (iii) we know that, if $B \in \sigma^{\text{op}}(A)$ and $\lambda I - B$ is not invertible then $\lambda I - A$ is not invertible (not even invertible at infinity). So suppose $\lambda I - B$ is

invertible. If $\lambda I - A$ is not invertible then there is nothing to prove. If also $\lambda I - A$ is invertible then, by Theorem 4.26 (ix), which applies since $B \in \sigma^{\text{op}}(A) \subset L(E, \mathcal{P})$ as $A \in BDO(E) \subset L(E, \mathcal{P})$, it follows that

$$\|(\lambda I - B)^{-1}\| = \frac{1}{\nu(\lambda I - B)} \leq \frac{1}{\nu(\lambda I - A)} = \|(\lambda I - A)^{-1}\|.$$

■

5.2.2 The Case $E = E^\infty(X)$

Now we will combine the results of Theorems 4.28 and 5.9. Recall that the set $UM(E)$ was introduced just before (and studied in) Lemma 5.4.

Corollary 5.11 *Consider $E = E^\infty(X)$ where X has a predual X^\triangleleft , and suppose $A = I - K \in BDO(E)$ is rich, has a preadjoint $A^\triangleleft \in L(E^\triangleleft)$ where $E^\triangleleft = E^1(X^\triangleleft)$, and that $K \in UM(E)$. Then the following statements are equivalent.*

- (a) *all limit operators of A are injective ($\alpha(A_h) = 0$ for all $A_h \in \sigma^{\text{op}}(A)$) and there is an S -dense subset, σ , of $\sigma^{\text{op}}(A)$ such that $\beta(A_h) = 0$ for all $A_h \in \sigma$;*
- (b) *all limit operators of A are injective ($\alpha(A_h) = 0$ for all $A_h \in \sigma^{\text{op}}(A)$) and there is an S -dense subset, σ , of $\sigma^{\text{op}}(A)$ such that $\alpha(A_h^\triangleleft) = 0$ for all $A_h \in \sigma$;*
- (c) *A is invertible at infinity;*
- (d) *A is Fredholm.*

Proof. Note first that, by Lemma 5.6, each $A_h \in \sigma^{\text{op}}(A)$ has a well-defined preadjoint $A_h^\triangleleft \in L(E^\triangleleft)$ so that statement (b) is well-defined; in fact, by Lemma 5.6, $\{A_h^\triangleleft : A_h \in \sigma^{\text{op}}(A)\} = \sigma^{\text{op}}(A^\triangleleft)$. Since always $\beta(A_h) \geq \alpha(A_h^\triangleleft)$ [86], clearly (a) \Rightarrow (b).

If (b) holds then, noting that property (i) of Lemma 5.2 implies that $\sigma^{\text{op}}(K)$ is uniformly Montel on (E, s) , applying Theorem 4.28, $\sigma^{\text{op}}(A)$ is uniformly bounded below, which implies that the range of each $A_h \in \sigma^{\text{op}}(A)$ is closed. This implies that $\beta(A_h) = \alpha(A_h^\triangleleft) = 0$ [86] for each $A_h \in \sigma$, so that (b) \Rightarrow (a) and each $A_h \in \sigma$ is surjective.

Applying Theorem 4.28 again, we see that all the elements of $\sigma^{\text{op}}(A)$ are invertible and their inverses are uniformly bounded. Applying Theorem 5.9 we conclude that (a) \Leftrightarrow (b) \Leftrightarrow (c).

The implication (c) \Rightarrow (d) follows from Theorem 5.9 (ii) with $C = I$ and $-K \in M(E)$ by property (iii) of Lemma 5.2. Finally, (d) \Rightarrow (c) is Theorem 5.9 b). ■

We note that Corollary 5.11, for operators satisfying the conditions of the corollary, reduces the problem of establishing Fredholmness and/or invertibility at infinity on $E^\infty(X)$ to one of establishing injectivity of the elements of $\sigma^{\text{op}}(A)$ and of a subset of $\sigma^{\text{op}}(A^\natural)$. In applications in mathematical physics this injectivity can sometimes be established directly via energy or other arguments (e.g. [30]), this reminiscent of classical applications of boundary integral equations in mathematical physics where $A = I + K$ with K compact, and injectivity of A is established from equivalence with a boundary value problem.

5.2.3 The One-Dimensional Case, $E = \ell^\infty(\mathbb{Z}, X)$

In the one-dimensional case, $N = 1$, a stronger version of Theorem 4.28 can be shown, namely Theorem 5.12 below. This result is shown by establishing, in the case in which $A = I - K \in BDO(E)$ is rich, $K \in UM(E)$, and all the limit operators of A are injective, the following three statements:

- a) If $B \in \sigma^{\text{op}}(A)$ has a surjective limit operator then B is surjective itself.
- b) Every $B \in \sigma^{\text{op}}(A)$ has a self-similar limit operator.
- c) Self-similar limit operators (of A , including those of B) are surjective.

Here we call $A \in L(E)$ *self-similar* if $A \in \sigma^{\text{op}}(A)$. Roughly speaking, we think of self-similar operators as containing a copy of themselves, at infinity.

The proofs of a) and c) above both heavily rely on Corollary 5.5 (which is a consequence of Theorem 4.33 and hence of Theorem 4.19).

Theorem 5.12 *Suppose that $E = \ell^\infty(\mathbb{Z}, X)$, that $A = I - K \in BDO(E)$ is rich, that $K \in UM(E)$, and that all the limit operators of A are injective. Then all elements of $\sigma^{\text{op}}(A)$ are invertible and their inverses are uniformly bounded.*

We will henceforth say that A is subject to the *Favard condition* (as is customary in e.g. [169, 170, 94, 95, 38]) if

$$\text{all limit operators of } A \text{ are injective on } E^\infty. \quad (\text{FC})$$

Remark 5.13 Rabinovich and Roch study Fredholmness and the Fredholm index of rich operators of the form $A = I - K \in BDO(E)$, where all matrix entries κ_{ij} of $[K]$ are compact operators on X . For operators $K \in S(E) \supset BDO(E)$, the latter is equivalent to $K \in M(E)$, as opposed to our condition $K \in UM(E) \subset M(E)$, where $\{\kappa_{ij} : i, j \in \mathbb{Z}\} \subset L(X)$ is collectively compact. □

Before we give the proof of Theorem 5.12, we state the following simplified version of Corollary 5.11 for the one-dimensional case, as a consequence of Theorem 5.12.

Corollary 5.14 *Suppose $E = \ell^\infty(\mathbb{Z}, X)$ where X has a predual X^\natural , and suppose $A = I - K \in BDO(E)$ is rich, has a preadjoint $A^\natural \in L(E^\natural)$ where $E^\natural = \ell^1(\mathbb{Z}, X^\natural)$, and that $K \in UM(E)$. Then the following statements are equivalent:*

- (FC) *all limit operators of A are injective;*
- (a) *all elements of $\sigma^{\text{op}}(A)$ are invertible & their inverses are uniformly bounded;*
- (b) *A is invertible at infinity;*
- (c) *A is Fredholm.*

Proof of Theorem 5.12.

Let $E = \ell^\infty(\mathbb{Z}, X)$ with a complex Banach space X . We will write $UM_\$(E)$ for the set of all rich operators in $UM(E)$. The set of limit operators A_h of an operator A with respect to all sequences h going to $\pm\infty$ is denoted by $\sigma_\pm^{\text{op}}(A)$, respectively. Then $\sigma^{\text{op}}(A) = \sigma_+^{\text{op}}(A) \cup \sigma_-^{\text{op}}(A)$ holds (see Lemma 4.23). Moreover, we put $P := P_{\{0,1,2,\dots\}}$ and $Q := I - P$.

We now break the proof of Theorem 5.12 down into the following three propositions.

Proposition 5.15 *Let $A \in I + UM_\$(E)$ and $B \in \sigma_\pm^{\text{op}}(A)$. If (FC) holds for A and if B has one surjective limit operator, $C \in \sigma_\pm^{\text{op}}(B)$ (with the same choice of $+$ or $-$ as for B), then B is surjective itself.*

Proof. Suppose, without loss of generality, that $B \in \sigma_+^{\text{op}}(A)$. Then $B = A_h$ for some sequence h of integers $h(1), h(2), \dots \rightarrow +\infty$. By our assumption, there exists a surjective $C \in \sigma_+^{\text{op}}(B)$. By [106, Corollary 3.97], we have that $C = A_{\tilde{h}}$ with some integer sequence $\tilde{h}(1), \tilde{h}(2), \dots \rightarrow +\infty$, and by Theorem 4.26 (vii) and Lemma 5.3 we know that $C \in I + UM_\$(E)$.

By passing to subsequences, if necessary, we can always arrange that $\tilde{h}(n-1) < h(n) < \tilde{h}(n)$ for all $n \geq 2$, with $\tilde{h}(n) - h(n) \rightarrow +\infty$ and $h(n) - \tilde{h}(n-1) \rightarrow +\infty$ as $n \rightarrow \infty$. Now, for every $n \in \mathbb{N}$, define $g_+(n) := \tilde{h}(n) - h(n) > 0$ and $g_-(n) := \tilde{h}(n-1) - h(n) < 0$, and put

$$\begin{aligned} A_n &:= V_{g_-(n)} Q C V_{-g_-(n)} + V_{g_+(n)} P C V_{-g_+(n)} \\ &\quad + V_{-h(n)} P_{\{\tilde{h}(n-1), \dots, \tilde{h}(n)-1\}} A V_{h(n)}. \end{aligned}$$

Our plan is now to check the conditions (a)–(e) of Corollary 5.5 with $B = A_h$ in place of A and with $\mathcal{B} = \sigma^{\text{op}}(A)$, in order to conclude that B is surjective.

(a) It is easy to see that $A_n \xrightarrow{\mathcal{P}} A_h = B$ since $V_{-h(n)}AV_{h(n)} \xrightarrow{\mathcal{P}} A_h$.

(b) Since C is invertible it is Fredholm of index zero. So also $D_1 := PCP + QCQ = C - PCQ - QCP$ is Fredholm of index zero since PCQ and QCP are compact for $C \in I + UM(E)$ (note that all entries of $C - I$ are compact operators and that C can be norm-approximated by band operators C' in which case both $PC'Q$ and $QC'P$ have only finitely many non-zero entries). We claim that the same is true for $D_2 := V_{g_-(n)}QCQV_{-g_-(n)} + V_{g_+(n)}PCPV_{-g_+(n)} + P_{\{g_-(n), \dots, g_+(n)-1\}}$ and every $n \in \mathbb{N}$. Indeed, since

$$\begin{aligned} \ker D_2 &= \{(\dots, u(-2), u(-1), 0, \dots, 0, u(0), u(1), \dots) : (u(i)) \in \ker D_1\}, \\ \text{im } D_2 &= \{(\dots, u(-2), u(-1), v(g_-(n)), \dots, v(g_+(n) - 1), u(0), u(1), \dots) \\ &\quad : (u(i)) \in \text{im } D_1, v(j) \in X\} \end{aligned}$$

hold with the zeros and $v(j)$'s in the positions $\{g_-(n), \dots, g_+(n) - 1\}$ of the sequence, respectively, we get that

$$\dim \ker D_2 = \dim \ker D_1 < \infty, \quad \text{codim im } D_2 = \text{codim im } D_1 < \infty$$

and hence D_2 is also Fredholm with the same index (namely zero) as D_1 . But this proves that

$$\begin{aligned} A_n &= D_2 + V_{g_-(n)}QCPV_{-g_-(n)} + V_{g_+(n)}PCQV_{-g_+(n)} \\ &\quad + V_{-h(n)}P_{\{\tilde{h}(n-1), \dots, \tilde{h}(n)-1\}}(A - I)V_{h(n)} \end{aligned}$$

is Fredholm of index zero since all of QCP , PCQ and $P_{\{\tilde{h}(n-1), \dots, \tilde{h}(n)-1\}}(A - I)$ are compact. So each A_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(A_n - I) \subseteq \mathcal{M}(A - I) \cup \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $A - I \in UM(E)$ by our premise and $C - I \in UM(E)$ by Lemma 5.3.

(d) Moreover, if $(k(m)) \subseteq \mathbb{Z}$ is arbitrary and $(n(m)) \subseteq \mathbb{N}$ is increasing then, since A and C are rich, there exist subsequences, denoted again by $(k(m))$ and $(n(m))$, and an operator D such that

$$V_{-k(m)}A_{n(m)}V_{k(m)} \xrightarrow{\mathcal{P}} D.$$

It is an easy exercise to check that D is either a translate of B or a limit operator of B (in particular it may be a translate or limit operator of C). In each of these cases D is a limit operator of A , and so $D \in \mathcal{B}$.

(e) Every $D \in \mathcal{B}$ is injective by assumption (FC).

We have seen that conditions (a)–(e) of Corollary 5.5 are satisfied with $\mathcal{B} := \sigma^{\text{op}}(A)$ and we therefore conclude that B is surjective. ■

A concept that is related to self-similar operators is that of a recurrent operator. An operator $C \in L(E)$ is called *recurrent* (generalising [122]) if, for every limit operator D of C , it holds that $\sigma^{\text{op}}(D) = \sigma^{\text{op}}(C)$. It is easy to see that, if C is recurrent, then

- all limit operators of C are self-similar;
- all limit operators of C are recurrent;
- the local operator spectra $\sigma_+^{\text{op}}(C)$ and $\sigma_-^{\text{op}}(C)$ coincide with $\sigma^{\text{op}}(C)$.

We also remark that, in the proof of the following proposition, we even show the slightly stronger result that every rich operator has a recurrent limit operator (namely the operator denoted by B' in the proof). It is not difficult to see that an element $\sigma^{\text{op}}(B)$ of the partially ordered set (\mathcal{A}, \supseteq) in the proof below is maximal iff B is recurrent.

Proposition 5.16 *Every rich operator $B \in L(E)$ has a self-similar limit operator C .*

Proof. Let

$$\mathcal{A} := \{ \sigma^{\text{op}}(B) : B \in \sigma^{\text{op}}(A) \}$$

which is a partially ordered set, equipped with the order ' \supseteq '. To be able to apply Zorn's lemma to \mathcal{A} , we have to check that its conditions are satisfied. So let \mathcal{B} be a totally ordered subset of \mathcal{A} , i.e.

$$\mathcal{B} := \{ \sigma^{\text{op}}(B) : B \in \sigma \}$$

for a subset $\sigma \subseteq \sigma^{\text{op}}(A)$, such that for any two $B_1, B_2 \in \sigma$, we either have $\sigma^{\text{op}}(B_1) \supseteq \sigma^{\text{op}}(B_2)$ or $\sigma^{\text{op}}(B_2) \supseteq \sigma^{\text{op}}(B_1)$.

On $Z := \sigma^{\text{op}}(A)$ we define the following family of seminorms. Let

$$\varrho_{2n-1}(T) := \|P_n T\|, \quad \varrho_{2n}(T) := \|T P_n\|$$

for $n = 1, 2, \dots$ and every $T \in Z$, and denote the topology that is generated on Z by $\{\varrho_1, \varrho_2, \dots\}$ by \mathcal{T} . By [106, Proposition 1.65] and since $\|T\| \leq \|A\|$ for every

$T \in Z$, convergence in (Z, \mathcal{T}) is equivalent to \mathcal{P} -convergence on Z . Also, since \mathcal{T} is generated by a countable family of seminorms, the topological space (Z, \mathcal{T}) is metrisable. Therefore, the \mathcal{P} -sequential compactness mentioned in Theorem 4.26 (v) is in fact \mathcal{P} -compactness, by which we mean compactness in (Z, \mathcal{T}) . In particular, Z itself and all elements of \mathcal{B} are compact sets in (Z, \mathcal{T}) .

Now put $\Sigma := \bigcap_{B \in \sigma} \sigma^{\text{op}}(B)$. We claim that Σ is nonempty. Conversely, suppose

$$\emptyset = \Sigma = \bigcap_{B \in \sigma} \sigma^{\text{op}}(B).$$

Then

$$\bigcup_{B \in \sigma} (Z \setminus \sigma^{\text{op}}(B)) = Z \setminus \bigcap_{B \in \sigma} \sigma^{\text{op}}(B) = Z \setminus \Sigma = Z$$

is an open cover of Z . Since Z is compact, there is a finite subset $\{B_1, \dots, B_n\}$ of σ such that

$$Z = \bigcup_{i=1}^n (Z \setminus \sigma^{\text{op}}(B_i)) = Z \setminus \bigcap_{i=1}^n \sigma^{\text{op}}(B_i)$$

so that $\bigcap_{i=1}^n \sigma^{\text{op}}(B_i) = \emptyset$. But that is impossible since $\{\sigma^{\text{op}}(B_1), \dots, \sigma^{\text{op}}(B_n)\}$ is a finite subchain of \mathcal{B} consisting of nonempty sets that contain one another.

So $\Sigma \neq \emptyset$. Take a

$$T \in \Sigma = \bigcap_{B \in \sigma} \sigma^{\text{op}}(B) \subseteq \sigma^{\text{op}}(A).$$

From Theorem 4.26 (iv) we know that $\sigma^{\text{op}}(B) \supseteq \sigma^{\text{op}}(T)$ for every $B \in \sigma$. So $\sigma^{\text{op}}(T) \in \mathcal{A}$ is an upper bound on the chain \mathcal{B} .

Now we can apply Zorn's lemma to \mathcal{A} and get that our partially ordered set (\mathcal{A}, \supseteq) has a maximal element, say $\sigma^{\text{op}}(B')$ with some $B' \in \sigma^{\text{op}}(A)$. Now pick any $C \in \sigma^{\text{op}}(B')$. From Theorem 4.26 (iv) we get $\sigma^{\text{op}}(B') \supseteq \sigma^{\text{op}}(C)$. But the maximality of $\sigma^{\text{op}}(B')$ means that $\sigma^{\text{op}}(B') = \sigma^{\text{op}}(C)$. So $C \in \sigma^{\text{op}}(B') = \sigma^{\text{op}}(C)$ is a self-similar limit operator of A . ■

Proposition 5.17 *If $C \in I + UM_{\S}(E)$ is self-similar and subject to (FC) then C is surjective.*

Proof. Since C is self-similar, there is a sequence $h = (h(n))_{n \in \mathbb{Z}}$ with $|h(n)| \rightarrow \infty$ and $V_{-h(n)} C V_{h(n)} \xrightarrow{\mathcal{P}} C$ as $n \rightarrow \infty$. Suppose, for simplicity of our notations, that $h(n) \rightarrow +\infty$ and $h(n) > 0$ for all $n \in \mathbb{N}$. (The argument is completely analogous if $h(n) \rightarrow -\infty$, where we can suppose that $h(n) < 0$ for all $n \in \mathbb{N}$.)

For every $n \in \mathbb{N}$, define $C_n \in BDO(E)$ by

$$(C_n u)(i) := (CV_{-\alpha h(n)} u)(\beta), \\ i = \alpha h(n) + \beta, \alpha \in \mathbb{Z}, \beta \in \{0, \dots, h(n) - 1\},$$

so that C_n commutes with $V_{h(n)}$.

We claim that this construction is such that Corollary 5.5 applies to C (in place of A) with $\mathcal{B} = \sigma^{\text{op}}(C)$ and therefore proves that C is surjective. So it remains to check that conditions (a)–(e) of Corollary 5.5 are satisfied.

(a) It holds that $C_n \xrightarrow{\mathcal{P}} C$. This can be seen as follows. Fix an arbitrary $m \in \mathbb{N}$. For every $D \in L(E)$, it is a simple consequence of the definition of the norm in E that

$$\|D\| = \sup_{i \in \mathbb{Z}} \|P_{\{ih(n), \dots, (i+1)h(n)-1\}} D\| \quad \text{for all } n \in \mathbb{N}.$$

Therefore, for every $n \in \mathbb{N}$, it holds that $\|P_m(C - C_n)\| = \sup_{i \in \mathbb{Z}} \gamma(m, n, i)$ with

$$\gamma(m, n, i) := \|P_{\{ih(n), \dots, (i+1)h(n)-1\}} P_m(C - V_{ih(n)} C V_{-ih(n)})\|, \quad i \in \mathbb{Z}.$$

But then it is clear that $\|P_m(C - C_n)\| \rightarrow 0$ as $n \rightarrow \infty$ since $\gamma(m, n, 0) = 0$,

$$\gamma(m, n, -1) = \|P_{\{-m, \dots, -1\}}(C - V_{-h(n)} C V_{h(n)})\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and $\gamma(m, n, i) = 0$ for all $i \in \mathbb{Z} \setminus \{0, -1\}$ as soon as $|h(n)| > m$.

Analogously, for every $n \in \mathbb{N}$, we have $\|(C - C_n)P_m\| = \sup_{i \in \mathbb{Z}} \delta(m, n, i)$ with

$$\delta(m, n, i) := \|P_{\{ih(n), \dots, (i+1)h(n)-1\}}(C - V_{ih(n)} C V_{-ih(n)})P_m\|, \quad i \in \mathbb{Z}.$$

To see that $\sup_{i \in \mathbb{Z}} \delta(m, n, i) \rightarrow 0$ as $n \rightarrow \infty$, note that $\delta(m, n, 0) = 0$,

$$\delta(m, n, -1) = \|P_{\{-h(n), \dots, -1\}}(C - V_{-h(n)} C V_{h(n)})P_m\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and, for all $i \in \mathbb{Z} \setminus \{0, -1\}$,

$$\begin{aligned} \delta(m, n, i) &= \|P_{\{ih(n), \dots, (i+1)h(n)-1\}}(C - V_{ih(n)} C V_{-ih(n)})P_m\| \\ &\leq 2 \sup_{S, T} \|P_T C P_S\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ by Proposition 3.61 and $C \in BDO(E)$, where the supremum in the last expression is taken over all sets $S, T \subset \mathbb{Z}$ with $\text{dist}(S, T) \geq h(n) - m$.

(b) By Theorem 5.37 below and $C_n V_{h(n)} = V_{h(n)} C_n$ we get that C_n is surjective if injective.

(c) Clearly,

$$\bigcup_{n=1}^{\infty} \mathcal{M}(C_n - I) \subseteq \mathcal{M}(C - I)$$

is collectively compact in $L(U)$ since $C - I \in UM(E)$.

(d) Let $(k(m)) \subseteq \mathbb{Z}$ be arbitrary and $(m(n)) \subseteq \mathbb{N}$ be monotonically increasing. Write each $k(m)$ as $\alpha(m)h(n(m)) + \beta(m)$ with $\alpha(m) \in \mathbb{Z}$ and $\beta(m) \in \{0, \dots, h(n(m)) - 1\}$. Then

$$\begin{aligned} D_m &:= V_{-k(m)} C_{n(m)} V_{k(m)} = V_{-\beta(m)} V_{-h(n(m))}^{\alpha(m)} C_{n(m)} V_{h(n(m))}^{\alpha(m)} V_{\beta(m)} \\ &= V_{-\beta(m)} C_{n(m)} V_{\beta(m)} \end{aligned}$$

holds for each $m \in \mathbb{N}$. If $(\beta(m))_{m \in \mathbb{N}}$ has a bounded subsequence then it even has a constant subsequence, of value $\gamma \in \mathbb{Z}$ say, and the corresponding subsequence of (D_m) converges to $V_{-\gamma} C V_{\gamma}$. Being a translate of $C \in \sigma^{\text{op}}(C) = \mathcal{B}$, this operator is also in $\sigma^{\text{op}}(C) = \mathcal{B}$. If $(\beta(m))_{m \in \mathbb{N}}$ goes to infinity, then, since C is rich, it has a subsequence for which the corresponding subsequence of (D_m) is \mathcal{P} -convergent to a limit operator of C , clearly also being an element of \mathcal{B} .

(e) All operators in $\mathcal{B} = \sigma^{\text{op}}(C)$ are injective by our assumption that (FC) holds for C . ■

5.3 Fredholmness in the Wiener Algebra

Again, let $E = E^p(X)$ with $p \in \{0\} \cup [1, \infty]$ and a Banach space X . If we write of “all spaces E ” in this section, we think of the family of spaces $E = E^p(X)$, where X is fixed and only p varies in $\{0\} \cup [1, \infty]$.

Recall from Section 3.7.3 that an operator

$$A = \sum_{k \in \mathbb{Z}^N} M_{b_k} V_k \tag{5.10}$$

with $b_k \in \ell^\infty(\mathbb{Z}^N, L(X))$ for all $k \in \mathbb{Z}^N$ is in the Wiener algebra $\mathcal{W}(E)$ iff

$$\|A\|_{\mathcal{W}} = \sum_{k \in \mathbb{Z}^N} \|b_k\|_{\infty} < \infty$$

and that such an operator is bounded, and in fact band-dominated, on all spaces E , which is why we often just write \mathcal{W} instead of $\mathcal{W}(E)$ (keeping in mind that N and X are given but p can vary). Instead, we will sometimes write \mathcal{W}_X when the underlying space X matters.

We have already seen that invertibility (hence spectrum), Fredholmness (hence essential spectrum) and Fredholm index of an operator $A \in \mathcal{W}$ do not depend on the space E that it is considered as acting on. Now we will pick up the results from the last section and concretise them for the Wiener algebra setting.

Lemma 5.18 [143, Proposition 2.5.6] *If $A \in \mathcal{W}$ is rich, as an operator on one space E , and if $h = (h(n)) \subset \mathbb{Z}^N$ tends to infinity then there is a subsequence g of h such that the limit operator A_g exists with respect to all spaces E . This limit operator again belongs to \mathcal{W} , and $\|A_g\|_{\mathcal{W}} \leq \|A\|_{\mathcal{W}}$.*

As a consequence of this lemma we get that the operator spectrum $\sigma^{\text{op}}(A)$ is contained in the Wiener algebra \mathcal{W} and does not depend on the underlying space E if $A \in \mathcal{W}$. So for $A \in \mathcal{W}$, the statement of Theorem 5.9 (iii) holds independently of the underlying space E . Moreover, by Theorem 5.9 (iv), also the uniform boundedness condition of the inverses is redundant for all p now since this is true for $p \in \{0, 1, \infty\}$ and consequently, by Riesz-Thorin interpolation, also for $p \in (1, \infty)$:

Theorem 5.19 *If $A \in \mathcal{W}$ is rich then the following statements are equivalent.*

- (i) *A is invertible at infinity on **one** of the spaces E .*
- (ii) *A is invertible at infinity on **all** the spaces E .*
- (iii) *All limit operators of A are invertible on **one** of the spaces E .*
- (iv) *All limit operators of A are invertible on **all** the spaces E and*

$$\sup_{p \in \{0\} \cup [1, \infty]} \sup_{A_h \in \sigma^{\text{op}}(A)} \|A_h^{-1}\|_{L(E^p)} < \infty. \quad (5.11)$$

Remark 5.20 This theorem is a strengthening and simplification of Theorem 2.5.7 in [143]. The theorem in [143] requires that X is reflexive, and, in the case that X is reflexive, it implies only a reduced version of our Theorem 5.19 with the value of E restricted to E^p , $p \in \{0\} \cup (1, \infty)$, in (i)–(iii). \square

Proof of Theorem 5.19. (i) \Rightarrow (iii) follows from Theorem 5.9 (iii).

(iii) \Rightarrow (iv): Suppose (iii) holds. We have observed already that $\sigma^{\text{op}}(A) \subset \mathcal{W}$ is independent of the space E by Lemma 5.18 (ii). Applying Lemma 5.18 (i) to the limit operators of A , it follows that these limit operators are invertible on all the spaces E . By Theorem 5.9 (iv),

$$s_p := \sup_{A_h \in \sigma^{\text{op}}(A)} \|A_h^{-1}\|_{L(E^p)}$$

is finite for $p \in \{0, 1, \infty\}$. Now, by Riesz-Thorin interpolation (as demonstrated in the proof of [143, Theorem 2.5.7]), we get that $s_p \leq s_1^{1/p} s_\infty^{1-1/p} < \infty$ for all $p \in (1, \infty)$, which proves (iv).

(iv) \Rightarrow (ii) follows from Theorem 5.9 (iii).

Finally, (ii) \Rightarrow (i) is evident. ■

From the above result and the relationship between invertibility at infinity and Fredholmness, (see Figure 4.1, Theorem 5.9 and Remark 5.10), we can deduce Corollary 5.22 below, which relates Fredholmness to invertibility of limit operators. In this corollary we require, for the equivalence of (a)-(d) with (e), the existence of a predual X^\diamond and, for A considered as an operator on E^∞ , the existence of a preadjoint $A^\diamond \in E^1(X^\diamond)$.

The following obvious lemma characterises existence of a preadjoint in terms of existence of preadjoints of the matrix entries of $[A]$.

Lemma 5.21 *If $A \in \mathcal{W}_X$, with $[A] = (a_{ij})$, and X has a predual X^\diamond , then A , considered as an operator on $E^\infty(X)$, has a preadjoint $A^\diamond \in E^1(X^\diamond)$ iff each entry $a_{ij} \in L(X)$ of the matrix representation of A has a preadjoint $a_{ij}^\diamond \in L(X^\diamond)$. If this latter condition holds then a preadjoint is $A^\diamond \in \mathcal{W}_{X^\diamond}$ with $[A^\diamond] = (a_{ji}^\diamond)$. In particular, A has a preadjoint if X is reflexive, given by $A^\diamond \in \mathcal{W}_{X^*} = \mathcal{W}_{X^\diamond}$, with $[A^\diamond] = (a_{ji}^*)$, where $a_{ji}^* \in L(X^*) = L(X^\diamond)$ is the adjoint of a_{ji} .*

Corollary 5.22 *Suppose $A = I - K \in \mathcal{W}$ is rich, and $K \in UM(E)$. Then the following statements are equivalent.*

- (a) *All limit operators of A are injective on E^∞ and $\sigma^{\text{op}}(A)$ has an S -dense subset of operators that are surjective on E^∞ ;*
- (b) *A is invertible at infinity on **all** the spaces E ;*
- (c) *A is invertible at infinity on **one** of the spaces E ;*
- (d) *A is Fredholm on **all** the spaces E .*

In the case that X has a predual X^\diamond and A , considered as an operator on $E^\infty(X)$, has a preadjoint $A^\diamond \in E^1(X^\diamond)$, then (a)-(d) are equivalent to

- (e) *A is Fredholm on **one** of the spaces E .*

Proof. For the clarity of our argument we introduce two more statements:

- (f) All limit operators of A are invertible on E^∞ ;
- (g) A is invertible at infinity on E^2 .

Each of these will turn out to be equivalent to (a)–(d).

By Theorem 4.28, statement (a) is equivalent to (f), which, by Theorem 5.19, is equivalent to each of (b), (c) and (g).

Since $K \in M(E)$, the implication (b) \Rightarrow (d) follows from Theorem 5.9 (ii) (applied with $C = -I$).

Since, obviously, (d) implies Fredholmness of A on E^2 , it also implies (g), by Theorem 5.9 (i). Another obvious consequence of (d) is (e).

Finally, suppose X^\natural and A^\natural exist and (e) holds for $E = E^p$. If $p = \infty$, then (c) follows by Theorem 5.9 b), and otherwise, if $p < \infty$, then (c) follows by Theorem 5.9 (i). ■

The above corollary implies, for rich operators in the Wiener algebra which are of the form $A = I - K \in \mathcal{W}$ with $K \in UM(E)$ (i.e. K is subject to the (equivalent) properties (i)–(v) in Lemma 5.2) and which possess a preadjoint, that Fredholmness on one of the spaces E implies Fredholmness on all spaces E . The argument to show this is indirect: it depends on the connection between Fredholmness and invertibility at infinity and on the equivalence of (i) and (ii) in Theorem 5.19. By the more direct approach in the proof of Theorem 3.68 above, one gets the same result, that Fredholmness on one of the spaces E implies Fredholmness on all spaces E , even without the condition that $A = I - K$ with $K \in UM(E)$ but for the price that X has to have the hyperplane property.

By combining Theorem 5.19 and Corollary 5.22 we get the following result.

Corollary 5.23 *Suppose $A = I - K \in \mathcal{W}$ is rich, $K \in UM(E)$, X has a predual X^\natural , and that A , considered as an operator on $E^\infty(X)$, has a preadjoint $A^\natural \in E^1(X^\natural)$. Then statements (i)–(iv) of Theorem 5.19 and (a)–(e) of Corollary 5.22 are all equivalent. Further, on every space E it holds that*

$$\text{spec}_{\text{ess}}(A) = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}(B). \quad (5.12)$$

Proof. It remains only to show that, for every $\lambda \in \mathbb{C}$, $\lambda I - A = (\lambda - 1)I + K$ is Fredholm iff $(\lambda - 1)I + L$ is invertible for every $L \in \sigma^{\text{op}}(K)$. For $\lambda \neq 1$ this follows from the earlier part of the corollary. This is true also for $\lambda = 1$ when X is finite-dimensional (see Corollary 5.24 below). When X is infinite-dimensional and $\lambda = 1$, then $(\lambda - 1)I + K = K \in UM(E)$ and, by Lemma 5.3,

also $(\lambda - 1)I + L = L \in UM(E)$. This implies that all the entries of the matrix representations of $(\lambda - 1)I + K$ and $(\lambda - 1)I + L$ are compact (i.e. in $K(X)$). Since X is infinite-dimensional, this implies that $(\lambda - 1)I + K$ and $(\lambda - 1)I + L$ are not Fredholm. ■

In the particularly simple case of a finite-dimensional space X , we have the following extended version of Corollary 5.23.

Corollary 5.24 *Suppose $A \in \mathcal{W}$ and X is finite-dimensional. Then statements (i)–(iv) of Theorem 5.19 and (a)–(e) of Corollary 5.22 are all equivalent. Moreover, if A is subject to all these equivalent statements then the index of A is the same on each space E . Further, on every space E , (5.12) holds.*

Proof. To see that the conditions of Corollary 5.23 are satisfied, recall Lemma 5.1 and remember that, by Lemma 5.4 (b), $K = I - A \in UM(E)$ if $\dim X < \infty$. Also recall that finite-dimensional spaces X are reflexive, so that $X^\natural = X^*$ is a predual and existence of a preadjoint follows from Lemma 5.21. The independence of the index follows from Theorem 3.68. ■

Finally, we note that in the one-dimensional case, $N = 1$, we have the following refinement of Corollaries 5.22, 5.23 and 5.24, as a consequence of Theorem 5.12.

Corollary 5.25 *Suppose $N = 1$ and that $A = I - K \in \mathcal{W}$ is rich and $K \in UM(E)$. Then the following statements are equivalent:*

- (FC) *All limit operators of A are injective on E^∞ ;*
- (a) *All limit operators of A are invertible on **one** of the spaces E ;*
- (b) *All limit operators of A are invertible on **all** the spaces E and (5.11) holds;*
- (c) *A is invertible at infinity on **all** the spaces E ;*
- (d) *A is invertible at infinity on **one** of the spaces E ;*
- (e) *A is Fredholm on **all** the spaces E .*

In the case that X has a predual X^\natural and A , considered as an operator on $E^\infty(X) = \ell^\infty(\mathbb{Z}, X)$, has a preadjoint A^\natural on $E^1(X^\natural) = \ell^1(\mathbb{Z}, X^\natural)$, then all of the above are equivalent to

- (f) *A is Fredholm on **one** of the spaces E ;*

and on every space E it holds that

$$\text{spec}_{\text{ess}}(A) = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}(B) = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}_{\text{point}}^{\infty}(B). \quad (5.13)$$

Here we denote by $\text{spec}_{\text{point}}^{\infty}(B)$ the point spectrum (set of eigenvalues) of B , considered as an operator on E^{∞} ; that is,

$$\text{spec}_{\text{point}}^{\infty}(B) = \{\lambda \in \mathbb{C} : (\lambda I - B)u = 0 \text{ for some nonzero } u \in E^{\infty}\}.$$

And finally, there is the case when X is finite-dimensional and $N = 1$:

Corollary 5.26 *Suppose $N = 1$, $A \in \mathcal{W}$ and X is finite-dimensional. Then statements (FC),(a)–(f) of Corollary 5.25 are equivalent. Moreover, if A is subject to all these equivalent statements then the index of A is the same on each space E . Further, on every space E , (5.13) holds.*

5.4 Limit Operators and the Fredholm Index

From Theorem 5.9 (also see Remark 5.10) we know that Fredholmness is the same as invertibility at infinity for band-dominated operators A on $E = E^p(X)$ if $\dim X < \infty$, whence it can be equivalently characterised in terms of the operator spectrum $\sigma^{\text{op}}(A)$. Interestingly, even the Fredholm index of A can be restored from its operator spectrum $\sigma^{\text{op}}(A)$ if also $N = 1$.

So in this section we will suppose that $N = 1$ and that X is finite-dimensional. We again abbreviate $P := P_{\mathbb{N}_0}$ and $Q := I - P$. Now let us suppose that $A \in BDO(E)$. Via the evident formula $E = \text{im } P \dot{+} \text{im } Q$, we can decompose A into four operators:

$$A = \left(\begin{array}{c|c} QAQ & QAP \\ \hline PAQ & PAP \end{array} \right) = \left(\begin{array}{c|c} \text{[diagonal block]} & \text{[off-diagonal block]} \\ \hline \text{[off-diagonal block]} & \text{[diagonal block]} \end{array} \right).$$

If $A \in BO(E)$, then the two blocks PAQ and QAP are finite-rank operators, whence they are compact if A is band-dominated. Consequently,

$$A = (P + Q)A(P + Q) = PAP + PAQ + QAP + QAQ$$

is Fredholm if and only if

$$A - PAQ - QAP = PAP + QAQ = \begin{pmatrix} \blacksquare & \\ & \blacksquare \end{pmatrix} \quad (5.14)$$

is Fredholm, where the two Fredholm indices coincide in this case. For the study of the operator (5.14), we put

$$A_+ := PAP + Q \quad \text{and} \quad A_- := QAQ + P, \quad (5.15)$$

which gives us that (5.14) equals the product $A_+A_- = A_-A_+$.

From this equality it follows that (5.14), and hence A , is Fredholm if and only if A_+ and A_- are Fredholm, and it holds that

$$\text{ind} A = \text{ind} A_+ + \text{ind} A_-. \quad (5.16)$$

Clearly,

$$\sigma^{\text{op}}(A_+) = \sigma_+^{\text{op}}(A) \cup \{I\} \quad \text{and} \quad \sigma^{\text{op}}(A_-) = \sigma_-^{\text{op}}(A) \cup \{I\},$$

whence the Fredholmness of A_{\pm} is determined by the local operator spectrum $\sigma_{\pm}^{\text{op}}(A)$, respectively. But also the index of A_{\pm} is hidden, in an astonishingly simple way, in $\sigma_{\pm}^{\text{op}}(A)$, respectively. Indeed, if we call

$$\text{ind}^+ A := \text{ind} A_+ = \text{ind}(PAP + Q)$$

and

$$\text{ind}^- A := \text{ind} A_- = \text{ind}(QAQ + P)$$

the *plus-* and the *minus-index* of the band-dominated operator A , then the following result holds.

Proposition 5.27 *Let $A \in BDO(E)$ be Fredholm, where $E = E^p$ with $1 < p < \infty$. Then all operators in $\sigma_+^{\text{op}}(A)$ have the same plus-index, and this number coincides with the plus-index of A . Analogously, all operators in $\sigma_-^{\text{op}}(A)$ have the same minus-index, and this number coincides with the minus-index of A .*

This remarkable result was derived by Rabinovich, Roch and ROE in [139] via computations of the K-group of the C^* -algebra $BDO(E^2)$, and it was generalized to $1 < p < \infty$ in [152]. It should be mentioned that the result of [152] extends

to operators in $BDO(E^1)$ and $BDO(E^\infty)$ that belong to the Wiener algebra \mathcal{W} , by our Theorem 3.68. Proposition 5.27 was re-proved by completely different techniques (using the sequence of the finite sections of A) in [146] and generalised to the case of an arbitrary Banach space X in case $A = I + K$ with an operator $K \in M(E)$ (i.e. all entries of $[K]$ are compact operators on X) in [138].

Corollary 5.28 *Let $A \in BDO(E)$ be Fredholm, where $E = E^p$ with $1 < p < \infty$. Then, for any two limit operators $B \in \sigma_+^{\text{op}}(A)$ and $C \in \sigma_-^{\text{op}}(A)$ of A , the identities*

$$\begin{aligned} \text{ind}^+ A &= \text{ind}^+ B = -\text{ind}^- B, \\ \text{ind}^- A &= \text{ind}^- C = -\text{ind}^+ C \quad \text{and} \\ \text{ind} A &= \text{ind}^+ B + \text{ind}^- C = \text{ind}(PBP + Q) + \text{ind}(QCQ + P) \end{aligned}$$

hold.

Proof. If A is Fredholm, then B and C are invertible by Theorem 5.9, whence $\text{ind} B$ and $\text{ind} C$ are both equal to zero. The rest is immediate from Proposition 5.27 and (5.16). ■

5.5 Different Types of Diagonal Behaviour

To allow more precise statements on the operator spectrum (and hence on Fredholmness and invertibility at infinity), we will now restrict ourselves to operators with diagonals/coefficients¹ in a particular subclass of $\ell^\infty(\mathbb{Z}^N, L(X))$. These classes will be the periodic, almost periodic, slowly oscillating and pseudoergodic functions, respectively.

5.5.1 Periodic and Almost Periodic Operators

Almost Periodic Sequences vs. Functions

Let Z be a Banach space. We say that a sequence $b = (b(n)) \in E^\infty(Z) = \ell^\infty(\mathbb{Z}^N, Z)$ is *periodic* if there are linearly independent vectors $\omega_1, \dots, \omega_N \in \mathbb{Z}^N$ such that

$$V_{\omega_k} b = b, \quad k = 1, \dots, N.$$

¹Looking at (3.20) or (5.10) and Remark 3.59, the phrases ‘diagonals of the matrix/operator’ and ‘coefficients of the operator’ are often used synonymously.

If we put $M := [\omega_1, \dots, \omega_n] \in \mathbb{Z}^{N \times N}$, then $S := M([0, 1]^N) \cap \mathbb{Z}^N$ is the discrete parallelotope spanned by $\omega_1, \dots, \omega_N$, and b is completely determined by the finitely many values $b(n)$ with $n \in S$. Consequently, the set

$$\{V_k b\}_{k \in \mathbb{Z}^N} = \{V_k b\}_{k \in S}$$

is finite and has (at most) $\#S = |\det M|$ elements. A slightly weaker property than periodicity is the following.

A sequence $b \in E^\infty(Z)$ is called *almost periodic* if the set $\{V_k b\}_{k \in \mathbb{Z}^N}$ is relatively compact in $E^\infty(Z)$. We write $E_{\text{AP}}^\infty(Z)$ for the set of all almost periodic sequences in $E^\infty(Z)$.

For $\varepsilon > 0$, we say that $k \in \mathbb{Z}^N$ is an ε -almost period of $b \in E^\infty(Z)$, if

$$\|V_k b - b\|_\infty < \varepsilon,$$

and we denote the set of all ε -almost periods of b by $\Omega_\varepsilon(b)$.

Proposition 5.29 *$b \in E^\infty(Z)$ is almost periodic if and only if, for every $\varepsilon > 0$, the set $\Omega_\varepsilon(b)$ is relatively dense in \mathbb{Z}^N ; that is, there exists a bounded set $S_\varepsilon \subset \mathbb{Z}^N$ such that every translate of S_ε in \mathbb{Z}^N contains an ε -almost period of b .*

Proof. The proof for $N = 1$ can be found in [28] (Theorem 1.26) for example. The result is easily carried over to arbitrary $N \in \mathbb{N}$. ■

As a consequence of Proposition 5.29, we get the following result (which is [106, Proposition 3.61]).

Proposition 5.30 *If $b \in E^\infty(L(X))$ is almost periodic, then*

$$\sigma_s^{\text{op}}(M_b) = \sigma^{\text{op}}(M_b)$$

for all $s \in S^{N-1}$.

Now we come to complex functions f of one or more real variables. If we identify a multiplication operator $\mathbf{M}_f : g \mapsto fg$ on $L^p = L^p(\mathbb{R}^N)$ with its counterpart on E via (2.4); that is a generalized multiplication operator on $E = \ell^p(\mathbb{Z}^N, X)$ with $X = L^p([0, 1]^N)$, we can carry over the notion of almost periodicity to L^∞ .

Definition 5.31 *We say that $f \in L^\infty$ is \mathbb{Z} -almost periodic if the set $\{V_k f\}_{k \in \mathbb{Z}^N}$ is relatively compact in L^∞ , and it is almost periodic if the set $\{V_k f\}_{k \in \mathbb{R}^N}$ is relatively compact in L^∞ . The sets of all \mathbb{Z} -almost and almost periodic functions in L^∞ will be denoted by L_{ZAP}^∞ and L_{AP}^∞ , respectively.*

It follows that

$$L_{\mathbf{AP}}^\infty \subset L_{\mathbb{Z}\mathbf{AP}}^\infty \subset L_{\mathbf{S}}^\infty,$$

where $L_{\mathbf{S}}^\infty$ denotes the set of all $f \in L^\infty$ for which the multiplication operator \mathbf{M}_f is rich (as an operator on $E \cong L^p$). The following example shows that $L_{\mathbf{AP}}^\infty$ is in fact a proper subset of $L_{\mathbb{Z}\mathbf{AP}}^\infty$.

Example 5.32 If $N = 1$ and $s > 0$, then

$$f(x) = (-1)^{[x/s]}, \quad x \in \mathbb{R}$$

is a step function with step size s . It is easily seen that $f \in L_{\mathbb{Z}\mathbf{AP}}^\infty$ if and only if $s \in \mathbb{Q}$. By [106, Proposition 3.35] it also follows that $f \in L_{\mathbf{S}}^\infty$ iff $s \in \mathbb{Q}$.

If we allow real-valued shift distances k in $\{V_k f\}$, then, for example, the sequence $(V_{n\sqrt{2}s}f)_{n \in \mathbb{N}}$ does not contain a convergent subsequence, regardless of the choice of s . So the set $\{V_k f\}_{k \in \mathbb{R}}$ is not relatively compact, whence $f \notin L_{\mathbf{AP}}^\infty$. \square

There is the following nice relationship between almost periodic functions and almost periodic sequences.

Lemma 5.33 *A sequence $b = (b(n)) \in \ell^\infty(\mathbb{Z}^N, \mathbb{C})$ is almost periodic if and only if there is a function $f \in L_{\mathbf{AP}}^\infty$ with $b(n) = f(n)$ for all $n \in \mathbb{Z}^N$.*

Proof. For $N = 1$, this is shown in Theorem 1.27 of [28]. But this construction easily generalizes to $N \geq 1$. \blacksquare

We prepare an equivalent characterization for functions in $L_{\mathbf{AP}}^\infty$. Therefore, for every $a \in \mathbb{R}^N$, put

$$f_a(x) := \exp(i \langle a, x \rangle), \quad x \in \mathbb{R}^N$$

where $\langle \cdot, \cdot \rangle$ refers to the standard inner product in \mathbb{R}^N . We denote by Π the set of all complex linear combinations of functions f_a with $a \in \mathbb{R}^N$, and we refer to elements of Π as trigonometric polynomials. Then the following holds.

Proposition 5.34 *For $f \in L^\infty = L^\infty(\mathbb{R}^N)$, the following are equivalent.*

- (i) *The set $\{V_k f\}_{k \in \mathbb{R}^N}$ is relatively compact in L^∞ (i.e. $f \in L_{\mathbf{AP}}^\infty$).*
- (ii) *f is continuous, and, for every $\varepsilon > 0$, there is a cube $S \subset \mathbb{R}^N$ such that every translate of S contains a $k \in \mathbb{R}^N$ with $\|V_k f - f\|_\infty < \varepsilon$.*
- (iii) *There is a sequence in Π which converges to f in the norm of L^∞ .*

Proof. For $N = 1$ this is a well-known result by Bohr and BOCHNER which can be found in any textbook on almost periodic functions, for example [28] or [87,

Chapter VI]. In fact, the equivalence of (i) and (iii) holds for locally compact abelian groups in place of \mathbb{R}^N (see [87], page 192). The equivalence of (i) and (ii) is written down for $N = 1$ in Chapter VI, Theorem 5.5 of [87] which literally applies to $N \geq 1$ as well. ■

As in the discrete case, the vectors k in (ii) are called the ε -almost periods of f . From (iii) we get that L_{AP}^∞ is the smallest Banach subalgebra of L^∞ that contains all functions f_a with $a \in \mathbb{R}^N$,

$$L_{\text{AP}}^\infty = \text{clos}_{L^\infty} \Pi = \text{closalg}_{L^\infty} \{f_a : a \in \mathbb{R}^N\}.$$

From this and the fact that $f_a \in BUC$ for all $a \in \mathbb{R}^N$, with BUC denoting the Banach algebra of all bounded and uniformly continuous functions $\mathbb{R}^N \rightarrow \mathbb{C}$, we get that

$$L_{\text{AP}}^\infty \subset BUC. \quad (5.17)$$

Finally, let $C(\overline{\mathbb{R}^N})$ stand for the Banach algebra of all continuous functions $f : \mathbb{R}^N \rightarrow \mathbb{C}$ for which $f(x)$ converges as $x \rightarrow \infty_s$ for all directions $s \in S^{N-1}$, and refer to the Banach subalgebra of L^∞ that is generated by L_{AP}^∞ and $C(\overline{\mathbb{R}^N})$ as

$$L_{\text{SAP}}^\infty := \text{closalg}_{L^\infty} \{L_{\text{AP}}^\infty, C(\overline{\mathbb{R}^N})\}.$$

The elements of L_{SAP}^∞ are called *semi-almost periodic functions*.

For $N = 1$, if $b_+ \in C(\overline{\mathbb{R}})$ with $b_+(+\infty) = 1$ and $b_+(-\infty) = 0$, and $b_- := 1 - b_+$ are fixed, then a famous result by Sarason [160] says that every function $f \in L_{\text{SAP}}^\infty$ has a unique representation of the form

$$f = b_- f_- + b_0 + b_+ f_+ \quad (5.18)$$

with $f_+, f_- \in L_{\text{AP}}^\infty$ and $b_0 \in C(\overline{\mathbb{R}})$ with $b_0(\pm\infty) = 0$.

In contrast to almost periodic functions, semi-almost periodic functions can have different almost periodic behaviour towards different directions of infinity. For dozens of very beautiful pictures of (semi-)almost periodic functions, see [20].

From (5.17) and $C(\overline{\mathbb{R}^N}) \subset BUC$, we get that

$$L_{\text{SAP}}^\infty \subset BUC \subset L_{\mathfrak{S}}^\infty.$$

From (5.18) and Proposition 5.30, we see that, for $N = 1$ and every $f \in L_{\text{SAP}}^\infty$,

$$\sigma_-^{\text{op}}(\mathbf{M}_f) = \sigma_-^{\text{op}}(\mathbf{M}_{f_-}) = \sigma^{\text{op}}(\mathbf{M}_{f_-}) \quad \text{and} \quad \sigma_+^{\text{op}}(\mathbf{M}_f) = \sigma_+^{\text{op}}(\mathbf{M}_{f_+}) = \sigma^{\text{op}}(\mathbf{M}_{f_+})$$

with $f_+, f_- \in L_{\text{AP}}^\infty$ uniquely determined by (5.18).

We now come to operators with almost periodic diagonals. For the first part we will even relax the restriction of being a band-dominated operator to merely boundedness on $E = E^p(X)$.

General Periodic and Almost Periodic Operators

Recall from Theorem 4.26 (v) that $A \in L(E)$ is a rich operator iff the set $\mathcal{T}(A)$ of all translates of A (recall (4.16)) is relatively \mathcal{P} -sequentially compact. The following is clearly a bit stronger: We call $A \in L(E)$ an *almost periodic operator* if the set $\mathcal{T}(A)$ is relatively compact in the norm topology on $L(E)$, and we call $A \in L(E)$ a *periodic operator* if every sequence in $\mathcal{T}(A)$ has a constant subsequence, i.e. iff $\mathcal{T}(A)$ is a finite set. It is easy to establish the following characterisation.

Lemma 5.35 *An operator $A \in L(E)$ is periodic iff there exist $m_1, \dots, m_N \in \mathbb{N}$ such that*

$$VA = AV \quad \text{for all} \quad V \in \tilde{\mathcal{V}}_A := \{V_{m_j e^{(j)}}\}_{j=1}^N$$

with $e^{(1)}, \dots, e^{(N)}$ denoting the standard unit vectors in \mathbb{R}^N , i.e. $e^{(j)}(i) = 1$ if $i = j$ and $= 0$ otherwise.

Example 5.36 We now continue the discussion from Example 4.14, where, due to the equalities $V_{-k}M_bV_k = M_{V_{-k}b}$ and $\|M_b\| = \|b\|$, we found out that a multiplication operator M_b with $b \in \ell^\infty(\mathbb{Z}^N, L(X))$ is rich iff the set

$$\{V_k b\}_{k \in \mathbb{Z}^N} \tag{5.19}$$

is relatively sequentially compact in the strict topology on $\ell^\infty(\mathbb{Z}^N, L(X))$. It can be shown moreover [143, Theorem 2.1.16] that this is the case iff the set $\{b(m) : m \in \mathbb{Z}^N\}$ is relatively compact in $L(X)$.

It is also clear that M_b is almost periodic iff the set (5.19) is relatively compact in the norm topology on $\ell^\infty(\mathbb{Z}^N, L(X))$, i.e. if b is almost periodic ($b \in E_{\text{AP}}^\infty(L(X))$).

Similarly, M_b is periodic iff every sequence in (5.19) has a constant subsequence, i.e. iff (5.19) is finite. By Lemma 5.35 this is equivalent to the requirement that there exist $m_1, \dots, m_N \in \mathbb{N}$ such that

$$b(k + m_j e^{(j)}) = b(k), \quad k \in \mathbb{Z}^N, \quad j = 1, \dots, N,$$

i.e. to the requirement that $b(k)$ is periodic as a function of each of the components of $k \in \mathbb{Z}^N$ (also see above). \square

Suppose that $A \in L(E^\infty)$ is periodic. For $n \in \mathbb{N}$, let

$$E_n^\infty = E_n^\infty(X) := \{u \in E^\infty(X) : V^n u = u \text{ for all } V \in \tilde{\mathcal{V}}_A\} \tag{5.20}$$

with $\tilde{\mathcal{V}}_A$ as defined in Lemma 5.35. (Note that E_n^∞ depends on A although we do not reflect this in the notation in order to keep notations as simple as possible.) Then E_n^∞ is a closed subspace of E^∞ consisting of periodic elements; $u \in E_n^\infty$ iff

$$u(k + n m_j e^{(j)}) = u(k), \quad k \in \mathbb{Z}^N, \quad j = 1, \dots, N,$$

where the integers m_1, \dots, m_N are as in the definition of $\tilde{\mathcal{V}}_A$ in Lemma 5.35. Clearly, $u \in E_n^\infty$ is determined by its components in the box

$$C_n := \{i = (i_1, \dots, i_N) \in \mathbb{Z}^N : -n \frac{m_j}{2} < i_j \leq n \frac{m_j}{2}, \quad j = 1, \dots, N\}.$$

Define the projection operator $\tilde{P}_n : E^\infty \rightarrow E_n^\infty$ by the requirement that

$$(\tilde{P}_n u)(k) = u(k) \quad \text{for all} \quad k \in C_n. \quad (5.21)$$

Then, clearly, for each n , $\tilde{P}_n Q_j = 0$ for all sufficiently large j , so that $\tilde{P}_n \in SN(E^\infty)$ by Lemma 3.2. (Note however that $\tilde{P}_n \notin L(E^\infty, \mathcal{P})$.)

The last part of the following result and its proof can be seen as a generalisation of Theorem 2.10 in [43]:

Theorem 5.37 *If $A \in L(E^\infty)$ is periodic then $A(E_n^\infty) \subset E_n^\infty$ for each n , and*

$$\sigma^{\text{op}}(A) = \{V_{-i} A V_i : i \in \mathbb{Z}^N\} = \{V_{-i} A V_i : i \in C_1\}. \quad (5.22)$$

If also $A = I + K$ with $K \in S(E^\infty) \cap M(E^\infty)$ and A is injective then A is invertible.

Proof. If $u \in E_n^\infty$ then $Au \in E_n^\infty$ since $V^n(Au) = AV^n u = Au$ for every $V \in \tilde{\mathcal{V}}_A$. From the definitions and Lemma 5.35 it is clear that (5.22) holds. Suppose now that $K \in S(E^\infty) \cap M(E^\infty)$ and that $A = I + K$ is injective. First we show that, for every n , $I + K\tilde{P}_n$ is invertible. To see injectivity, suppose $u \in E^\infty$ and $(I + K\tilde{P}_n)u = 0$. Then $u = -K\tilde{P}_n u \in E_n^\infty$ since $\tilde{P}_n u \in E_n^\infty$ and $K = A - I$ is periodic. Now $u \in E_n^\infty$ implies $\tilde{P}_n u = u$ and therefore $0 = (I + K\tilde{P}_n)u = (I + K)u = Au$, i.e. $u = 0$ by injectivity of A . Now surjectivity of $I + K\tilde{P}_n$ follows from its injectivity by the Riesz theory for compact operators in topological vector spaces [151] since $K\tilde{P}_n \in KS(E^\infty)$ by Lemma 3.23.

Next, note that from (5.22) it follows that $A \in \sigma^{\text{op}}(A)$ and that, since A is injective, all the limit operators of A are injective. Further, by (5.22), it follows that $\sigma^{\text{op}}(K)$ is uniformly Montel since $K \in M(E^\infty)$. Applying Theorem 4.28 we see that the limit operators of A are uniformly bounded below, in particular that A is bounded below.

To see finally that A is surjective let $v \in E^\infty$ and set $v_n = \tilde{P}_n v \in E_n^\infty$ and $u_n = (I + K\tilde{P}_n)^{-1}v_n$ so that $u_n + K\tilde{P}_n u_n = v_n$ which implies (as seen above) that $u_n \in E_n^\infty$, that $\tilde{P}_n u_n = u_n$, and hence that $Au_n = u_n + Ku_n = v_n$. Since (v_n) is bounded and A is bounded below, also (u_n) is bounded. Since $K \in M(E^\infty)$ it follows that there exists an $u \in E^\infty$ and a subsequence of (u_n) , denoted again by (u_n) , such that $Ku_n \xrightarrow{s} v - u$, so that $u_n = v_n - Ku_n \xrightarrow{s} v - (v - u) = u$. As $K \in S(E^\infty)$ this implies $Ku_n \xrightarrow{s} Ku$ so that $v - u = Ku$ and hence $Au = u + Ku = v$. ■

The above result has the following obvious corollary, phrased in the spirit of Theorem 4.28.

Corollary 5.38 *If $A = I + K \in S(E^\infty)$ is periodic and $K \in M(E^\infty)$ and if all limit operators of A are injective, then all limit operators of A are invertible (with uniformly bounded inverses).*

Further down, in Theorem 5.44, we will see that, in the case when A is also band-dominated, this corollary holds more generally with ‘periodic’ replaced by ‘almost periodic’; indeed in the one-dimensional case $N = 1$ we have seen in Theorem 5.12 that this corollary holds even with ‘periodic’ replaced by ‘rich’. We conjecture that the latter holds in all dimensions $N \in \mathbb{N}$; at least we know of no examples where the requirement for an S -dense subset of surjective operators in Theorem 4.28 is not redundant.

We continue with a collection of results for the general setting of all almost periodic operators $A \in L(E)$; a set that shall be denoted by $AP(E)$ for brevity. Our first result follows from a slightly more general result which is Theorem 6.5.2 in Kurbatov [95]. An alternative proof is given in [39, Lemma 6.9].

Lemma 5.39 *$AP(E)$ is an inverse closed Banach subalgebra of $L(E)$.*

We now summarise a number of results on almost periodic operators and their limit operators, some of which can also be found in [144].

Theorem 5.40 *For $A \in AP(E)$, the following holds.*

(i) *If, for some sequence $h = (h(1), h(2), \dots) \in \mathbb{Z}^N$ and $B \in L(E)$,*

$$V_{-h(n)} A V_{h(n)} \xrightarrow{\mathcal{P}} B \quad \text{holds, then} \quad V_{-h(n)} A V_{h(n)} \rightrightarrows B.$$

(ii) *$A \in \sigma^{\text{op}}(A)$ (i.e. A is self-similar).*

- (iii) $\sigma^{\text{op}}(A) = \text{clos}_{L(E)}\mathcal{T}(A)$ is a compact subset of $AP(E)$.
- (iv) A is invertible iff any one of its limit operators is invertible.
- (v) $\nu(A) = \nu(B)$ for all $B \in \sigma^{\text{op}}(A)$, so that A is bounded below iff $\sigma^{\text{op}}(A)$ is uniformly bounded below.
- (vi) If u is almost periodic then Au is almost periodic.
- (vii) If A is invertible on E^∞ then it is invertible on E_{AP}^∞ .

Proof. (i) Since $A \in AP(E)$, every subsequence of $V_{-h(n)}AV_{h(n)}$ ($\xrightarrow{\mathcal{P}} B$) has a norm-convergent subsequence the limit of which must be B . But this proves norm convergence of the whole sequence.

(ii) Let $h(n) = (n^2, 0, \dots, 0) \in \mathbb{Z}^N$ for every $n \in \mathbb{N}$. Since $A \in AP(E)$, there is a subsequence g of h such that $V_{-g(n)}AV_{g(n)}$ converges. But then

$$\|V_{-(g(n+1)-g(n))}AV_{g(n+1)-g(n)} - A\| = \|V_{-g(n+1)}AV_{g(n+1)} - V_{-g(n)}AV_{g(n)}\| \rightarrow 0$$

as $n \rightarrow \infty$, showing that $A = A_f \in \sigma^{\text{op}}(A)$ with $f(n) = g(n+1) - g(n) \rightarrow \infty$.

(iii) The inclusion $\sigma^{\text{op}}(A) \subset \text{clos}_{L(E)}\mathcal{T}(A)$ follows from (i). The reverse inclusion follows from (ii), from Theorem 4.26 (ii) and the closedness of $\sigma^{\text{op}}(A)$ (see Theorem 4.26 (iii) above or [106, Corollary 3.96]). The compactness of $\text{clos}_{L(E)}\mathcal{T}(A)$ follows from the relative compactness of $\mathcal{T}(A)$ in $L(E)$. By Lemma 5.39, every operator in $\text{clos}_{L(E)}\mathcal{T}(A)$ is almost periodic.

(iv) Take an arbitrary limit operator A_h of A and let $h = (h(1), h(2), \dots) \subset \mathbb{Z}^N$ be such that $A_n := V_{-h(n)}AV_{h(n)} \xrightarrow{\mathcal{P}} A_h$ holds. By (i) we have that $A_n \rightrightarrows A_h$. If A_h is invertible, then so is A_n for every large n , and therefore A is invertible. Conversely, if A is invertible, then A_h is invertible by a basic result on Banach algebras (see e.g. [106, Lemma 1.3]) since $\|A_n^{-1}\| = \|A^{-1}\|$ is bounded.

(v) If $B \in \sigma^{\text{op}}(A)$, then, by (i), we have that $V_{-h(n)}AV_{h(n)} \rightrightarrows B$ for some sequence $h(1), h(2), \dots$ in \mathbb{Z}^N . By (4.10) this implies that $\nu(V_{-h(n)}AV_{h(n)}) \rightarrow \nu(B)$ as $n \rightarrow \infty$. On the other hand, since every $V_{h(n)}$ is an isometry, we have that

$$\nu(V_{-h(n)}AV_{h(n)}) = \inf_{\|u\|=1} \|V_{-h(n)}AV_{h(n)}u\| = \inf_{\|v\|=1} \|Av\| = \nu(A)$$

for every $n \in \mathbb{N}$, so that $\nu(A) = \nu(V_{-h(n)}AV_{h(n)}) \rightarrow \nu(B)$, i.e. $\nu(A) = \nu(B)$.

(vi) Let $h = (h(1), h(2), \dots) \subset \mathbb{Z}^N$ be arbitrary. If $A \in AP(E^\infty)$ and $u \in E_{\text{AP}}^\infty$ then there is a subsequence g of h such that both $V_{g(n)}AV_{-g(n)}$ and $V_{g(n)}u$ converge in the norm of $L(E^\infty)$ and E^∞ , respectively. But then also

$$V_{g(n)}(Au) = (V_{g(n)}AV_{-g(n)})(V_{g(n)}u)$$

converges in E^∞ , which shows that $Au \in E_{\text{AP}}^\infty$.

(vii) If $A \in AP(E^\infty)$ is invertible on E^∞ , then, by Lemma 5.39, also $A^{-1} \in AP(E^\infty)$. Now (vi) shows that $u \in E_{\text{AP}}^\infty$ iff $Au \in E_{\text{AP}}^\infty$. ■

Almost Periodic Band-Dominated Operators

We will now look at operators that are almost periodic and band-dominated at the same time. We first show that every almost periodic band-dominated operator can be approximated in the norm by almost periodic band operators. The same statement holds with ‘almost periodic’ replaced by ‘rich’, as was first pointed out in [104, Proposition 2.9]. The proof of our lemma is very similar to that of this related statement.

Remark 5.41 Recall from Remark 3.63 that the norm-approximation of a given band-dominated operator A by a sequence A_n of band operators is in general a more involved problem than it might seem. For a given $A \in BDO(E)$, in the proof of [143, Theorem 2.1.6] a sequence of band operators

$$A_n = \sum_{|k| \leq n} c_{k,n} B_k, \quad n \in \mathbb{N}, \quad (5.23)$$

$$\text{with } B_k = \int_{[0, 2\pi]^N} M_{e_t} A M_{e_{-t}} e^{-i(t,k)} dt, \quad k \in \mathbb{Z}^N \quad (5.24)$$

is constructed, where $c_{k,n} \in \mathbb{C}$ and $e_t(m) = e^{i(t_1 m_1 + \dots + t_N m_N)}$ for all $m \in \mathbb{Z}^N$ and $t \in \mathbb{R}^N$. This construction is such that each matrix $[B_k]$ is only supported on the k th diagonal and $A_n \rightrightarrows A$ as $n \rightarrow \infty$.

An alternative, less constructive method for approximating an almost periodic operator by an almost periodic band operator is described in step I of the proof of [94, Theorem 1]. □

Lemma 5.42 *For every band-dominated operator A and the corresponding approximating sequence (A_n) of band operators (5.23), the following holds.*

- (i) *If A is almost periodic, then each one of the band operators A_n is almost periodic.*
- (ii) *If A is rich and $\sigma^{\text{op}}(A)$ is uniformly Montel, then every operator spectrum $\sigma^{\text{op}}(A_n)$ is uniformly Montel.*

Proof. Since the integrand in (5.24) is continuous in t , the integral can be understood in the Riemann sense and therefore B_k can be approximated in norm by

the corresponding Riemann sums

$$R_m^{(k)} = \left(\frac{2\pi}{m}\right)^N \sum_{j=1}^{m^N} M_{e_{t_{m,j}}} A M_{e_{-t_{m,j}}} e^{-i(t_{m,j},k)}, \quad m \in \mathbb{N} \quad (5.25)$$

as $m \rightarrow \infty$. Here $t_{m,j} \in T_{m,j}$ are arbitrary where $\{T_{m,j} : j = 1, \dots, m^N\}$ is a partition of $[0, 2\pi]^N$ into hyper-cubes of width $2\pi/m$ (also see the proof of [143, Theorem 2.1.18]).

To prove (i) it suffices, by Lemma 5.39, to show that all Riemann sums $R_m^{(k)}$ are almost periodic. Since, as the restriction of an almost periodic (even periodic) function $\mathbb{R}^N \rightarrow \mathbb{C}$ to the integer grid \mathbb{Z}^N , the sequence e_t is almost periodic for every choice of $t \in \mathbb{R}^N$, we get that both $M_{e_{t_{m,j}}}$ and $M_{e_{-t_{m,j}}}$ are almost periodic (see Lemma 5.43 below). By Lemma 5.39 again and $A \in AP(E)$, it follows that then all of the Riemann sums $R_m^{(k)}$ and consequently, all operators B_k and A_n are almost periodic as well.

For the proof of (ii), let A be rich and $\sigma^{\text{op}}(A)$ be uniformly Montel. By Lemma 4.30, using that A is rich, we get that $A \in UM(E)$. Since every $M_{e_{t_{m,j}}}$ is rich (even almost periodic), we get that $M_{e_{t_{m,j}}} A M_{e_{-t_{m,j}}} \in UM(E)$ for all $m \in \mathbb{N}$ and $j \in \{1, \dots, m^N\}$, by Lemma 5.4 (c) and (d). This fact, together with formulas (5.23)–(5.25) and Lemma 5.4 (a), shows that all $R_m^{(k)}$, B_k and A_n are in $UM(E)$. But the latter implies that $\sigma^{\text{op}}(A_n)$ is uniformly Montel, by Lemma 4.30. ■

Lemma 5.43 *A band operator is almost periodic iff all of its diagonals are almost periodic; that means*

$$A = \sum_{k \in D} M_{b_k} V_k \in AP(E) \quad \text{iff} \quad b_k \in E_{\text{AP}}^\infty(L(X)), \quad \forall k \in D$$

for all finite sets $D \subset \mathbb{Z}^N$ and $b_k \in E^\infty(L(X))$, $k \in D$.

Proof. Let $D \subset \mathbb{Z}^N$ be finite, let $b_k \in E^\infty(L(X))$ for all $k \in D$, and put $A = \sum_{k \in D} M_{b_k} V_k$. Note that

$$V_{-m} A V_m = \sum_{k \in D} M_{V_{-m} b_k} V_k \quad (5.26)$$

for every $m \in \mathbb{Z}^N$. We show that a sequence of operators (5.26) converges in the operator norm iff all of the corresponding diagonals $V_{-m} b_k$ converge in the norm of $E^\infty(L(X))$.

Suppose $A \in AP(E)$ and take an arbitrary sequence $h = (h(n))_{n \in \mathbb{N}} \subset \mathbb{Z}^N$. Then there exists a subsequence g of h such that $V_{-g(n)} A V_{g(n)} \rightrightarrows C$ for some

$C \in L(E)$. Then, for all $i, j \in \mathbb{Z}^N$, with $[C] = [c_{i,j}]$ and with the restriction and extension operators R_i and E_j introduced in Section 3.6, we have that

$$\begin{aligned} \|V_{-g(n)}b_{i-j}(i) - c_{i,j}\|_{L(X)} &= \|R_i(V_{-g(n)}AV_{g(n)} - C)E_j\|_{L(X)} \\ &\leq \|V_{-g(n)}AV_{g(n)} - C\|_{L(E(X))} \rightarrow 0 \end{aligned} \quad (5.27)$$

as $n \rightarrow \infty$. Now, for every $k \in \mathbb{Z}^N$, define $c_k \in E^\infty(L(X))$ by $c_k(i) = c_{i,i-k}$, so that c_k is the k -th diagonal of C . From (5.27) we get that $\|V_{-g(n)}b_k - c_k\|_{E^\infty} \rightarrow 0$, so that $b_k \in E_{\text{AP}}^\infty(L(X))$ for each $k \in \mathbb{Z}^N$.

Now, conversely, suppose that $b_k \in E_{\text{AP}}^\infty(L(X))$ for all $k \in D$ and take an arbitrary sequence $h = (h(n))_{n \in \mathbb{N}}$. Let $\{k_1, k_2, \dots, k_m\}$ be an enumeration of $D \subset \mathbb{Z}^N$, and choose a subsequence $h^{(1)} \subset h$ such that $V_{-h^{(1)}(n)}b_{k_1}$ converges. From this choose a subsequence $h^{(2)} \subset h^{(1)}$ such that also $V_{-h^{(2)}(n)}b_{k_2}$ converges, etc., until we arrive at a sequence $g := h^{(m)} \subset h$ for which all $V_{-g(n)}b_k$ with $k \in D$ converge. Denote the respective limits by $c_k \in E^\infty(L(X))$. Then we have $V_{-g(n)}AV_{g(n)} \rightrightarrows \sum_{k \in D} M_{c_k}V_k =: C$ since

$$\begin{aligned} \|V_{-g(n)}AV_{g(n)} - C\|_{L(E)} &= \left\| \sum_{k \in D} (M_{V_{-g(n)}b_k} - M_{c_k})V_k \right\|_{L(E)} \\ &\leq \sum_{k \in D} \|V_{-g(n)}b_k - c_k\|_\infty \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, showing that $A \in AP(E)$. ■

Here is the announced generalisation of Corollary 5.38 to almost periodic band-dominated operators (for a proof using the above results and the approximation in the strict topology of almost periodic by periodic functions, see Section 6.4 in [39]). We note that results of this flavour in concrete cases, in particular showing something close to equivalence of (i) and (iii), date back at least to Shubin [170] for scalar elliptic differential operators with smooth almost periodic coefficients. Note also that we have already seen, in Theorem 5.12, that in the one-dimensional case, $N = 1$, the equivalence of (iii) and (iv) holds even when K is only rich rather than almost periodic.

Theorem 5.44 a) *If $A = I + K \in BDO(E^\infty)$ with $K \in UM(E^\infty)$ almost periodic then the following statements are equivalent.*

- (i) A is invertible;
- (ii) A is bounded below;
- (iii) all limit operators of A are injective, i.e. (FC) holds;

(iv) all limit operators of A are invertible with uniformly bounded inverses.

b) If the above holds and A is periodic then (i)–(iv) are also equivalent to

(v) A is injective.

5.5.2 Slowly Oscillating Operators

We now come to a class of operators with particularly simple limit operators. If we ask ourselves for which sequences $b \in \ell^\infty(\mathbb{Z}^N, L(X))$ it is true that all limit operators of M_b have a constant main diagonal (i.e. $(M_b)_h = M_c$ with $c = (a)_{n \in \mathbb{Z}^N}$ and $a \in L(X)$) the first class that comes to mind is all convergent sequences; that is, $b = (b(n))_{n \in \mathbb{Z}^N}$ for which $b(n)$ converges in $L(X)$ as $|n| \rightarrow \infty$. But this is only half the truth.

A sequence $b \in E^\infty(Z) = \ell^\infty(\mathbb{Z}^N, Z)$ with a Banach space Z is called *slowly oscillating* if, for all $k \in \mathbb{Z}^N$ (or, which is equivalent, for all $k \in \{-1, 0, 1\}^N$),

$$\|b(n+k) - b(n)\|_Z \rightarrow 0 \quad \text{as} \quad |n| \rightarrow \infty. \quad (5.28)$$

For example, if $Z = \mathbb{C}$ and $N = 1$ then $b(n) = \sin \sqrt{|n|}$ is an example of a non-convergent but slowly oscillating sequence. We write $E_{\text{SO}}^\infty(Z)$ for the set of all slowly oscillating sequences with values in Z . It is a standard exercise to check that $E_{\text{SO}}^\infty(Z)$ is a Banach subalgebra of $E^\infty(Z)$.

Property (5.28) is equivalent to $V_k b - b \in c_0(\mathbb{Z}^N, Z)$ for all $k \in \mathbb{Z}^N$ (or just $k \in \{-1, 0, 1\}^N$), which, on the other hand, is equivalent to each of the following

$$V_{-k} M_b V_k - M_b \in K(E, \mathcal{P}) \quad \text{and} \quad [M_b, V_k] \in K(E, \mathcal{P}).$$

The above observation generalises from multiplication operators $A = M_b$ to arbitrary operators $A \in BDO(E)$ with slowly oscillating diagonals. Keeping in mind that (see e.g. [143, Proposition 2.4.4]) a band-dominated operator is in $K(E, \mathcal{P})$ iff all of its diagonals are in $c_0(\mathbb{Z}^N, L(X))$, one gets the following result.

Proposition 5.45 *The following are equivalent for an operator $A \in BDO(E)$.*

- (i) $V_{-k} A V_k - A \in K(E, \mathcal{P})$ for all $k \in \mathbb{Z}^N$;
- (ii) $V_{-k} A V_k - A \in K(E, \mathcal{P})$ for all $k \in \{-1, 0, 1\}^N$;
- (iii) $[A, V_k] \in K(E, \mathcal{P})$ for all $k \in \mathbb{Z}^N$;
- (iv) $[A, V_k] \in K(E, \mathcal{P})$ for all $k \in \{-1, 0, 1\}^N$;

(v) *All diagonals of A are slowly oscillating.*

We call $A \in BDO(E)$ a *slowly oscillating operator* and write $A \in SO(E)$ if A is subject to the equivalent properties (i)–(v).

As we promised earlier, $A \in SO(E)$ has very simple limit operators. Indeed, let $B = A_h$ for some sequence $h = (h(n))_{n \in \mathbb{N}} \subset \mathbb{Z}^N$ going to infinity, then, by

$$(V_{-k}AV_k - A)_h = V_{-k}A_hV_k - A_h = V_{-k}BV_k - B, \quad k \in \mathbb{Z}^N \quad (5.29)$$

and the fact that $C_h = 0$ if $C \in K(E, \mathcal{P})$, we get that (5.29) is zero, so that $B = V_{-k}BV_k$ for all $k \in \mathbb{Z}^N$. So every limit operator B is what one calls a *translation invariant operator*; it has constant diagonals and is therefore a Laurent operator (recall Example 3.62) – but now with operator entries.

In a sense, the converse is true as well:

Proposition 5.46 *Let $A \in BDO(E)$ be rich. Then $A \in SO(E)$ iff all limit operators of A are translation invariant.*

Proof. If $A \in SO(E)$ then (even if A is not rich) all limit operators of A are translation invariant. Conversely, suppose $A \in BDO(E)$ is rich but not in $SO(E)$. Then one, say the j th, diagonal b of A is not slowly oscillating. By (5.28) there is an $\varepsilon > 0$, a $k \in \mathbb{Z}^N$ and a sequence $n_1, n_2, \dots \in \mathbb{Z}^N$ with $|n_i| \rightarrow \infty$ so that

$$\|b(n_i + k) - b(n_i)\| \geq \varepsilon, \quad i = 1, 2, \dots \quad (5.30)$$

Since A is rich there is a subsequence g of $h = (n_1, n_2, \dots)$ such that A_g exists. Since b is the j th diagonal of A , also $(M_b)_g =: M_c$ exists (see [143, Lemma 2.4.3]) and c is the j th diagonal of A_h . By (5.30) it follows that $\|c(k) - c(0)\| \geq \varepsilon$ so that not all diagonals of A_h are constant and therefore A_h is not translation invariant. (In fact, $V_{-k}A_hV_k \neq 0$.) ■

Note that slowly oscillating operators need not be rich if $\dim X = \infty$; but of course (see Lemma 5.1) they are rich if $\dim X < \infty$.

Proposition 5.47 *Let $E = E^\infty(X)$ and $A = I + K \in SO(E)$ be rich with $K \in M(E)$.*

- a) *Every limit operator of A is invertible if injective.*
- b) *A is invertible at infinity iff all limit operators of A are injective.*

Proof. a) Let B be a limit operator of A . Then B is translation invariant and hence periodic in the sense of Section 5.5.1. Now Theorem 5.37 shows that injectivity of B implies its invertibility.

b) This follows from part a) and Theorem 5.9 (iii) and (iv). ■

In the general setting $E = E^p(X)$ and without the condition that $A - I \in M(E)$, one still has the following result by Rabinovich and Roch [136] (and see [143, Theorem 2.4.27]) which says that the uniform boundedness condition in Theorem 5.9 (iii) is redundant for arbitrary values of p if A is slowly oscillating:

Proposition 5.48 *If $A \in SO(E)$ then A is invertible at infinity iff all limit operators of A are invertible.*

Apart from the above results, another pleasant fact about slowly oscillating operators A is that invertibility of their limit operators, i.e. Laurent operators $B = A_h$, can be checked effectively which leads to easily verifiable criteria for Fredholmness / invertibility at infinity of A and hence to nice and explicit formulas for the essential spectrum of A .

5.5.3 Pseudoergodic Operators

If slowly oscillating operators are those with especially simple limit operators then the operators we consider now, in some sense show the opposite behaviour.

Fix a Banach space Z and a bounded and closed subset Σ of Z . In accordance with Davies [51], we say that $b = (b(n)) \in \ell^\infty(\mathbb{Z}^N, Z)$ is *pseudoergodic with respect to Σ* if, for every finite set $S \subset \mathbb{Z}^N$, every function $c : S \rightarrow \Sigma$ and every $\varepsilon > 0$, there is a $k \in \mathbb{Z}^N$ such that

$$\max_{n \in S} \|b(n+k) - c(n)\|_Z < \varepsilon. \quad (5.31)$$

A little thought reveals that there are in fact infinitely many vectors $k \in \mathbb{Z}^N$ with (5.31) since this property also holds for all continuations of c to sets $S' \subset \mathbb{Z}^N$ containing S . Also it follows immediately from this definition that, if b is pseudoergodic with respect to Σ , then b is pseudoergodic with respect to every closed subset of Σ .

Roughly speaking, the sequence $b = (b(n))$ is pseudoergodic with respect to Σ if one can find, up to a given precision $\varepsilon > 0$, any finite pattern of elements from Σ somewhere in b . For example, it is conjectured² that the decimal expansion of $\pi = 3, 1415926535\dots$ forms a pseudoergodic sequence $\mathbb{N} \rightarrow \Sigma = \{0, \dots, 9\}$.

Example 5.49 – Random sequences. Pseudoergodicity is a deterministic concept that is designed to capture essential aspects of random behaviour. As an

²Go to <http://pi.nersc.gov/> to see if your name is coded in the first 4 billion digits of π .

example, let $\Sigma \subset Z$ be bounded and closed and let V be a random variable with values in Σ such that

$$p(\sigma, \varepsilon) := \mathbb{P}(\|V - \sigma\|_Z < \varepsilon) > 0$$

for all $\sigma \in \Sigma$ and $\varepsilon > 0$.

If $b = (b(n))_{n \in \mathbb{Z}^N}$ is a sequence of independent samples $b(n)$ from V then, with probability 1, b is pseudoergodic w.r.t. Σ . The argument is sometimes called “the Infinite Monkey Theorem” and it follows from the 2nd Borel Cantelli Lemma (see [27, Theorem 8.16] or [48, Theorem 4.2.4]).

Indeed, let $S \subset \mathbb{Z}^N$ be finite, $c : S \rightarrow \Sigma$ be arbitrary and take $\varepsilon > 0$. For every $k \in \mathbb{Z}^N$,

$$p := \mathbb{P}\left(\sup_{n \in S} \|b(n+k) - c(n)\|_Z < \varepsilon\right) = \prod_{n \in S} p(c(n), \varepsilon) > 0.$$

Consequently, for every $m \in \mathbb{N}$,

$$p_m := \mathbb{P}\left(\bigvee_{k \in \{-m, \dots, m\}^N} \left(\sup_{n \in S} \|b(n+k) - c(n)\|_Z < \varepsilon\right)\right) = 1 - (1-p)^{s_m},$$

where s_m denotes the number of vectors $k \in \mathbb{Z}^N$ for which $k + S \subset \{-m, \dots, m\}^N$. So finally,

$$\mathbb{P}\left(\bigvee_{k \in \mathbb{Z}^N} \left(\sup_{n \in S} \|b(n+k) - c(n)\|_Z < \varepsilon\right)\right) \geq p_m$$

for all $m \in \mathbb{N}$, whence the latter probability must be one since $p_m \rightarrow 1$ as $m \rightarrow \infty$.

Note that the argument can be adjusted to the case of random samples $b(n)$ that are not fully correlated (rather than independent). \square

It is easily seen that, if b is pseudoergodic with respect to Σ , then the operator spectrum of M_b contains every multiplication operator M_c one can think of with a Σ -valued sequence c . But also the reverse implication is true.

Proposition 5.50 *Let X be a Banach space, Σ be a bounded and closed subset of $L(X)$, and suppose $b \in \ell^\infty(\mathbb{Z}^N, L(X))$. Then b is pseudoergodic with respect to Σ if and only if*

$$\sigma^{\text{op}}(M_b) \supset \{M_c : c = (c(n))_{n \in \mathbb{Z}^N} \subset \Sigma\}. \quad (5.32)$$

Proof. It is readily seen that, if $A = M_b$ with $b = (b(n)) \in \ell^\infty(\mathbb{Z}^N, L(X))$ and $h = (h(m)) \subset \mathbb{Z}^N$ tends to infinity, then A_h exists and is equal to M_c with $c = (c(n))_{n \in \mathbb{Z}^N} \subset \Sigma$ if and only if

$$\sup_{n \in S} \|b(n + h(m)) - c(n)\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty \quad (5.33)$$

for all finite sets $S \subset \mathbb{Z}^N$.

Let b be pseudoergodic w.r.t. Σ , and take an arbitrary $c = (c(n))_{n \in \mathbb{Z}^N} \subset \Sigma$. For every $m \in \mathbb{N}$, define $h(m) \in \mathbb{Z}^N$ as the value of k in (5.31), where we put $Z = L(X)$, $S = \{-m, \dots, m\}^N$ and $\varepsilon = 1/m$. Since there are infinitely many choices for this vector k , we can moreover suppose that $|h(m)| > m$. Then $h = (h_m)$ converges to infinity, and it is easily seen that (5.33) holds for every bounded $S \subset \mathbb{Z}^N$, showing that $M_c \in \sigma^{\text{op}}(M_b)$.

Conversely, if $M_c \in \sigma^{\text{op}}(M_b)$ for every $c = (c(n))_{n \in \mathbb{Z}^N} \subset \Sigma$, then (5.33) holds for every finite set $S \subset \mathbb{Z}^N$. But this clearly implies that b is pseudoergodic with respect to Σ . ■

Clearly, for b to be pseudoergodic with respect to a set Σ , it is necessary that Σ is contained in the closure of the set of all components of b .

Lemma 5.51 *If $b = (b(n)) \in \ell^\infty(\mathbb{Z}^N, Z)$ is pseudoergodic with respect to $\Sigma \subset Z$, then $\Sigma \subset \text{clos}_Z\{b(n)\}_{n \in \mathbb{Z}^N}$.*

Proof. If b is pseudoergodic w.r.t. Σ and $a \in \Sigma$ then (5.31) with $S = \{0\}$ and $c(0) = a$ shows that, for every $\varepsilon > 0$, there is a $k \in \mathbb{Z}^N$ with $\|b(n) - a\|_Z < \varepsilon$. ■

We will say that $b = (b(n)) \in \ell^\infty(\mathbb{Z}^N, Z)$ is *pseudoergodic* if it is pseudoergodic with respect to $\Sigma = \text{clos}_Z\{b(n)\}_{n \in \mathbb{Z}^N}$, which is the largest possible choice for Σ .

Corollary 5.52 *Let X be a Banach space, and suppose $b \in \ell^\infty(\mathbb{Z}^N, L(X))$. Then b is pseudoergodic if and only if*

$$\sigma^{\text{op}}(M_b) = \{M_c : c = (c(n))_{n \in \mathbb{Z}^N} \subset \Sigma\} \quad (5.34)$$

with $\Sigma = \text{clos}_{L(X)}\{b(n)\}_{n \in \mathbb{Z}^N}$.

Proof. What remains to be shown is that the reverse implication holds as well in (5.32). But this follows from Lemma 4.27 e) with $w = 0$ and the choice of Σ . ■

We now come to band-dominated operators with the property analogous to (5.34). If

$$A = \sum_{k \in \mathbb{Z}^N} M_{b_k} V_k$$

is in $BDO(E)$ with $b_k \in \ell^\infty(\mathbb{Z}^N, L(X))$ for all $k \in \mathbb{Z}^N$ then we put

$$\Sigma_k := \text{clos}_{L(X)} \{b_k(n)\}_{n \in \mathbb{Z}^N}, \quad k \in \mathbb{Z}^N$$

and say that A is a *pseudoergodic operator* if

$$\sigma^{\text{op}}(A) = \left\{ \sum_{k \in \mathbb{Z}^N} M_{c_k} V_k : c_k = (c_k(n))_{n \in \mathbb{Z}^N} \subset \Sigma_k, k \in \mathbb{Z}^N \right\}.$$

By this definition, we automatically get that $A \in \sigma^{\text{op}}(A)$ holds, i.e. pseudoergodic operators are self-similar.

Using similar arguments as above one can see that every diagonal b_k of a pseudoergodic operator A is pseudoergodic. The converse is a bit more involved:

It is not true that A is pseudoergodic if all its diagonals are pseudoergodic. To see the latter look at $A = M_b + M_b V_1$ with $N = 1$ and $b \in \ell^\infty(\mathbb{Z}, L(X))$ pseudoergodic and note that $\sigma^{\text{op}}(A)$ only contains operators with both diagonals identical to each other. However, if $D = \{k_1, \dots, k_m\} \subset \mathbb{Z}^N$ is a finite set and

$$A = \sum_{k \in D} M_{b_k} V_k$$

is a band operator with diagonals $b_k \in \ell^\infty(\mathbb{Z}^N, L(X))$, $k = k_1, \dots, k_m$ then A is pseudoergodic if

$$\left((b_{k_1}(n), \dots, b_{k_m}(n)) \right)_{n \in \mathbb{Z}^N}$$

is pseudoergodic w.r.t. $\Sigma = \Sigma_{k_1} \times \dots \times \Sigma_{k_m}$, where $\Sigma_k = \text{clos}_{L(X)} \{b_k(n)\}_{n \in \mathbb{Z}^N}$ for all $k \in D$.

For example, if A is a band operator with each diagonal a random sequence in the sense of Example 5.49 then, almost surely, A is pseudoergodic.

In a universe of infinite size, anything that has a non-zero probability of occurring must occur infinitely often. Thus at any instant of time – for example, the present moment – there must be an infinite number of identical copies of each of us doing precisely what each of us is now doing. There are also infinite numbers of identical copies of each one of us doing something other than what we are doing at this moment. Indeed, an infinite number of copies of each of us could be found at this moment doing anything that it was possible for us to do with a non-zero probability at this moment.

5.6 Comments and References

Characterising Fredholmness in terms of the limit operators starts in [123] (for semi-Fredholmness) and then builds up through [96, 140] (for $\dim X < \infty$ and $1 < p < \infty$), [141] (for $E = L^2(\mathbb{R})$), [143] (now we have $\dim X = \infty$ but still $p \in (1, \infty)$) and [113, 104, 106] (and now also $p \in [1, \infty]$ and $\dim X = \infty$) to the results stated in this chapter, where the latest improvements are from [38, 39] and [107] (see Remark 5.10 and Sections 5.2.2, 5.2.3 and 5.3).

The first successful computation of the Fredholm index of a general band-dominated operator via limit operator techniques was given (by means of K-theory) by Rabinovich, Roch and Roe in [139] for the case $p = 2$, $N = 1$ and $X = \mathbb{C}$. The result was later carried over (using standard Fredholm arguments) to $p \in (1, \infty)$ by Roch [152] and to $p \in [1, \infty]$ by Chandler-Wilde and the author [39, 107]. In the meantime Rabinovich, Roch and Silbermann [146] proved the same formula for the Fredholm index of A by arguments involving the finite sections of A . Related more recent results are also in [162, 163].

We have not mentioned the limit operator approaches of e.g. [81] to the approximation of spectra and pseudospectra for rather general band-dominated operators. Much more can be said in this direction in the case of particular classes of band-dominated operators, for example, see [18] for the case of banded Toeplitz operators.

Chapter 6

Stable Approximation of Infinite Matrices

Hagen, Roch and Silbermann begin their monograph [81] with the following neat overview:

- **Functional Analysis:** Solve equations in infinitely many variables.
- **Linear Algebra:** Solve equations in finitely many variables.
- **Numerical Analysis:** Build the bridge!

The latter is done by approximation methods. We now present the rather general concept of approximation methods, slightly adapted to our case at hand: the sequence space $E = E^p(X)$.

6.1 Approximation Methods

6.1.1 Definitions

If $A \in L(E)$ is an invertible operator then the equation

$$Au = b \tag{6.1}$$

has a unique solution $u \in E$ for every right-hand side $b \in E$. We will deal with the approximate solution of this equation where $A \in L(E)$ and $b \in E$ are given and $u \in E$ is to be determined.

For this purpose, let $(E_n)_{n \in \mathbb{N}}$ refer to a sequence of Banach subspaces of E which are the images of projection operators $\Pi_n : E \rightarrow E$ and which exhausts E

in the sense that the projections Π_n are \mathcal{P} -convergent to the identity operator I on E as $n \rightarrow \infty$.

If one has to solve an equation of the form (6.1), one tries to approximate, again in the sense of \mathcal{P} -convergence, the operator $A \in L(E)$ by a sequence of operators $(\tilde{A}_n)_{n \in \mathbb{N}}$ with $\tilde{A}_n \in L(E_n)$ and to solve the somewhat simpler equations

$$\tilde{A}_n \tilde{u}_n = \Pi_n b \quad (6.2)$$

in E_n instead, hoping these are uniquely solvable (at least for all sufficiently large n) if (6.1) is uniquely solvable and that the solutions $\tilde{u}_n \in E_n$ of (6.2) tend to the solution $u \in E$ of (6.1) in the strict topology as n goes to infinity. If this is the case for every right-hand side $b \in E$ then we say that the *approximation method* (\tilde{A}_n) is *applicable* to A .

This is the idea of approximation methods – or, to be more precise, of projection methods. It is somewhat unsatisfactory that every operator of the sequence (\tilde{A}_n) acts on a different space. To overcome this difficulty so that we may regard A and all operators of the approximation method as acting on the same space E , we will henceforth identify $\tilde{A}_n \in L(E_n)$ with $A_n := \tilde{A}_n + \Theta_n \in L(E)$, where $\Theta_n := I - \Pi_n$ is the complementary projector of Π_n . Then, for every $n \in \mathbb{N}$, (6.2) is equivalent to

$$A_n u_n = b \quad \text{alias} \quad \begin{pmatrix} \tilde{A}_n & 0 \\ 0 & \Theta_n \end{pmatrix} \begin{pmatrix} \tilde{u}_n \\ \Theta_n b \end{pmatrix} = \begin{pmatrix} \Pi_n b \\ \Theta_n b \end{pmatrix} \quad (6.3)$$

with respect to the decomposition $E = E_n \dot{+} \text{im } \Theta_n$. The approximation method (6.2) is applicable to A if and only if (6.3) is so. Clearly, \tilde{A}_n and A_n are simultaneously invertible, where $\|A_n^{-1}\| = \max(\|\tilde{A}_n^{-1}\|, 1)$.

Recall that we call a sequence of operators (A_n) *stable* if there is a $n_0 \in \mathbb{N}$ such that A_n is invertible for all $n \geq n_0$, and

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

Consequently, the sequence (\tilde{A}_n) is stable (where every operator \tilde{A}_n is acting on a different space E_n) if and only if the sequence (A_n) is.

A quite natural and very popular choice of the sequence (A_n) is

$$A_n := \Pi_n A \Pi_n + \Theta_n, \quad n \in \mathbb{N} \quad (6.4)$$

which we call the *natural projection method* for the operator A and the sequence (E_n) of Banach spaces.

We will next make the subspaces E_n more precise. Therefore, take a monotonously increasing sequence $(\Omega_n)_{n \in \mathbb{N}}$ of finite subsets of \mathbb{Z}^N which exhausts \mathbb{Z}^N in the sense that, for every $k \in \mathbb{Z}^N$, there is a $n_0 \in \mathbb{N}$ with $k \in \Omega_n$ for all $n \geq n_0$.

Then put

$$\Pi_n = P_{\Omega_n}, \quad \Theta_n = Q_{\Omega_n}, \quad \text{and} \quad E_n := \text{im } \Pi_n.$$

In this setting, we call (6.4) the *finite section method* and write $A_{[n]}$ for the operator (6.4).

Example 6.1 Let $N = 1$, and put $\Omega_n = \{-n, \dots, n\}$ for every $n \in \mathbb{N}$. Then, in matrix language, the infinite system (6.1); that is

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & a_{-1,-1} & a_{-1,0} & a_{-1,1} & \cdots \\ \cdots & a_{0,-1} & a_{0,0} & a_{0,1} & \cdots \\ \cdots & a_{1,-1} & a_{1,0} & a_{1,1} & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ u(-1) \\ u(0) \\ u(1) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ b(-1) \\ b(0) \\ b(1) \\ \vdots \end{pmatrix},$$

is replaced by the sequence of finite quadratic systems (6.2), namely the truncations

$$\begin{pmatrix} a_{-n,-n} & \cdots & a_{-n,n} \\ \vdots & & \vdots \\ a_{n,-n} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} \tilde{u}_n(-n) \\ \vdots \\ \tilde{u}_n(n) \end{pmatrix} = \begin{pmatrix} b(-n) \\ \vdots \\ b(n) \end{pmatrix},$$

for $n = 1, 2, \dots$. This is where the name ‘finite section method’ comes from.

During this procedure one is keeping fingers crossed that the latter systems are uniquely solvable once they are sufficiently large, and that $\tilde{u}_n(k) \rightarrow u(k)$ as $n \rightarrow \infty$ for every $k \in \mathbb{Z}$. \square

Remark 6.2 In the case of a space E of functions in a continuous variable, like $E = L^p(\mathbb{R}^N) \cong \ell^p(\mathbb{Z}^N, L^p([0, 1]^N))$, and an operator A acting on E it can make sense to study approximation sequences $(A_n)_{n \in \mathbb{I}}$ to A with a continuous index set like $\mathbb{I} = \mathbb{R}_+ = (0, +\infty)$ instead of $\mathbb{I} = \mathbb{N}$.

In [103, 104, 106] it is demonstrated that much of the following can be generalised to the setting of a continuous index set \mathbb{I} . But since we want to keep our exposition here simple instead of spoiling it with unpleasant technicalities, we restrict ourselves to the case $\mathbb{I} = \mathbb{N}$ and refer to [106, §1.7, §2.4] for the more general case. \square

6.1.2 Applicability vs. Stability

We will see that the question whether an approximation method $(A_n)_{n \in \mathbb{N}}$ is applicable or not heavily depends on the stability of the sequence (A_n) . Recall that we call a sequence (A_n) *stable* if there is an index n_0 such that all operators A_n with $n \geq n_0$ are invertible and their inverses are uniformly bounded.

The classic Polski theorem [81] says that a **strongly** convergent approximation method (A_n) is applicable to A – with convergence of the solutions u_n to u in $(E, \|\cdot\|)$ – if and only if the operator A is invertible and the sequence (A_n) is stable. It was proven by Roch and Silbermann in [154] (also see Theorem 6.1.3 in [143]) that the same is true for the applicability of the methods that we have in mind: **\mathcal{P} –convergent** methods (A_n) with solutions u_n convergent in (E, s) – provided that the sequence (A_n) is subject to the following condition:

Write $(A_n) \in \mathcal{F}(E, \mathcal{P})$ if the sequence (A_n) is bounded, and, for every $k \in \mathbb{N}$,

$$\sup_{n \in \mathbb{N}} \|P_k A_n Q_m\| \rightarrow 0 \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|Q_m A_n P_k\| \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \quad (6.5)$$

Of course, therefore it is necessary that $A_n \in L(E, \mathcal{P})$ for every $n \in \mathbb{N}$, and moreover $A \in L(E, \mathcal{P})$ by Proposition 3.48 if $A_n \xrightarrow{\mathcal{P}} A$.

Theorem 6.3 *If $(A_n) \in \mathcal{F}(E, \mathcal{P})$ is \mathcal{P} –convergent to A then the approximation method (A_n) is applicable to A if and only if A is invertible and (A_n) is stable.*

Proof. See Theorem 6.1.3 of [143] or [154] or 4.41f in [133]. ■

In the above setting the uniformity condition (6.5) is actually redundant:

Lemma 6.4 *If $(A_n)_{n \in \mathbb{N}} \subset L(E, \mathcal{P})$ is \mathcal{P} –convergent to A then $(A_n) \in \mathcal{F}(E, \mathcal{P})$.*

Proof. The boundedness of the sequence follows from Lemma 3.45. It remains to prove (6.5). Therefore fix an arbitrary $k \in \mathbb{N}$. From $(A_n) \subset L(E, \mathcal{P})$ we conclude that for every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there is a $m(n, \varepsilon) \in \mathbb{N}$ such that $\|P_k A_n Q_m\| < \varepsilon$ for all $m > m(n, \varepsilon)$. What we have to show is that

$$\forall \varepsilon > 0 \quad \exists m(\varepsilon) : \quad \|P_k A_n Q_m\| < \varepsilon \quad \forall m > m(\varepsilon) \quad \forall n \in \mathbb{N}. \quad (6.6)$$

So take an arbitrary $\varepsilon > 0$ and choose $m_0 \in \mathbb{N}$ large enough that $\|P_k A Q_m\| < \varepsilon/2$ for all $m > m_0$, which is possible since $A \in L(E, \mathcal{P})$ by Proposition 3.48. Moreover, choose $n_0 \in \mathbb{N}$ large enough that $\|P_k(A_n - A)\| < \varepsilon/2$ for all $n > n_0$.

For $m > m_0$ and $n > n_0$ we conclude

$$\|P_k A_n Q_m\| \leq \|P_k A Q_m\| + \|P_k(A_n - A)\| \|Q_m\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now choose $m(\varepsilon) := \max(m(1, \varepsilon), \dots, m(n_0, \varepsilon), m_0)$ to ensure (6.6). Analogously, we prove the second property in (6.5). ■

From Theorem 6.3 and Lemma 6.4 we immediately get the following.

Corollary 6.5 *If $(A_n)_{n \in \mathbb{N}} \subset L(E, \mathcal{P})$ is \mathcal{P} -convergent to A then the approximation method (A_n) is applicable to A if and only if A is invertible and (A_n) is stable.*

6.1.3 Stability vs. Invertibility at Infinity

In the previous section we have seen that one of the main ingredients to the applicability of an approximation method is its stability. The aim of this section is to translate the stability problem for a given approximation method into the question whether or not an associated operator is invertible at infinity.

Throughout the following, let again $E = E^p(X) = \ell^p(\mathbb{Z}^N, X)$ with $N \in \mathbb{N}$, $p \in [1, \infty]$ and X a complex Banach space, and put $E' = \ell^p(\mathbb{Z}^{N+1}, X)$, respectively. To make a distinction between operators on E and operators on E' , we will write I' , P'_U , P'_n , Q'_U , Q'_n for the identity operator and the respective projection operators on E' .

Stacked Operators

Definition 6.6 *Given an $u \in E'$, we define a sequence $(u_n)_{n \in \mathbb{Z}}$ with $u_n \in E$ by*

$$u_n(m) := u(m, n), \quad m \in \mathbb{Z}^N, n \in \mathbb{Z},$$

and regard u_n as the n -th layer of u .

In this sense we will henceforth think of elements $u \in E'$ as being composed by their layers $u_n \in E$, $n \in \mathbb{Z}$, where we will treat the whole sequence $(u_n)_{n \in \mathbb{Z}}$ as one object in

$$\ell^p(\mathbb{Z}, E) \cong \ell^p(\mathbb{Z}^{N+1}, X) = E'. \quad (6.7)$$

We will use the layer construction from Definition 6.6 to associate an operator on E' with every bounded sequence $(A_n)_{n \in \mathbb{Z}}$ of operators on E , simply by “stacking” this operator sequence to a “pile”.

Definition 6.7 *If $\mathbb{I} \subset \mathbb{Z}$ and $(A_n)_{n \in \mathbb{I}}$ is a bounded sequence of operators on E , then by*

$$(Bu)(m, n) := \begin{cases} (A_n u_n)(m) & \text{if } n \in \mathbb{I}, \\ u_n(m) & \text{if } n \notin \mathbb{I}, \end{cases} \quad m \in \mathbb{Z}^N, n \in \mathbb{Z},$$

an operator B on E' is given, which acts as A_n in the n -th layer of u for $n \in \mathbb{I}$, and it is the identity operator otherwise. In this sense, we will regard A_n as the n -th layer of B , and we will refer to B as the stacked operator of the sequence $(A_n)_{n \in \mathbb{I}}$, denoted by

$$\bigoplus_{n \in \mathbb{I}} A_n$$

or simply by $\bigoplus A_n$ if $\mathbb{I} = \mathbb{N}$.

By its definition, $\bigoplus A_n$ acts on every layer of $u \in E'$ independently from the other layers, like a generalised multiplication operator; indeed, if we identify u and (u_n) in the sense of (6.7) then we can identify $\bigoplus A_n$ with the operator of multiplication by a function in $\ell^\infty(\mathbb{Z}, L(E))$. By this identification, it is clear that the stacked operator is a bounded linear operator on E' with

$$\left\| \bigoplus_{n \in \mathbb{I}} A_n \right\|_{L(E')} = \begin{cases} s & \text{if } \mathbb{I} = \mathbb{Z}, \\ \max(s, 1) & \text{if } \mathbb{I} \neq \mathbb{Z}, \end{cases} \quad \text{where} \quad s = \sup_{n \in \mathbb{I}} \|A_n\|_{L(E)}.$$

Another simple observation is the following:

Lemma 6.8 *Let $(A_n)_{n \in \mathbb{N}} \subset L(E)$ be a bounded sequence. The operator $\bigoplus A_n$ is invertible in $L(E')$ if and only if the set $\{A_n\}_{n \in \mathbb{N}}$ is uniformly invertible.*

Here and in what follows we call a set of bounded linear operators *uniformly invertible* if all its elements are invertible and their inverses are uniformly bounded.

In general it is not true that $\bigoplus A_n \in BO(E)$ or $BDO(E)$ or $L(E', \mathcal{P}')$ whenever all operators A_n are of that kind.

Example 6.9 Take $A_n = V_n$ for all $n \in \mathbb{N}$. Then $(A_n) \subset BO(E)$ for all n but it is readily seen that $\bigoplus A_n$ is not even in $L(E', \mathcal{P}') \supset BDO(E') \supset BO(E')$. \square

But one can prove that $\bigoplus A_n$ is in $BO(E')$ or $BDO(E')$ or $L(E', \mathcal{P}')$ if all A_n are in the respective class, and, in addition, some uniformity condition is satisfied. To formulate this condition, we put (cf. [106]), for every $A \in L(E)$ and every non-negative integer d ,

$$f_A(d) := \sup_{U, V} \|P_V A P_U\|, \quad (6.8)$$

where the supremum is taken over all $U, V \subset \mathbb{Z}^N$ with $\text{dist}(U, V) \geq d$. We know from Proposition 3.61 that $A \in BDO(E)$ iff $f_A(d) \rightarrow 0$ as $d \rightarrow \infty$.

Proposition 6.10 *Suppose $(A_n)_{n \in \mathbb{N}} \subset L(E)$ is a bounded sequence.*

a) *If $(A_n)_{n \in \mathbb{N}} \subset BO(E)$ and the sequence of band-widths of A_n is bounded, i.e. $A_n \in BO_w(E)$ for some global $w \in \mathbb{N}$, then $\bigoplus A_n \in BO(E')$.*

b) If $(A_n)_{n \in \mathbb{N}} \subset BDO(E)$ and the condition

$$\sup_{n \in \mathbb{N}} f_{A_n}(d) \rightarrow 0 \quad \text{as} \quad d \rightarrow \infty$$

holds, then $\oplus A_n \in BDO(E')$.

c) If $(A_n)_{n \in \mathbb{N}} \subset L(E, \mathcal{P})$ and condition (6.5) holds; that is $(A_n) \in \mathcal{F}(E, \mathcal{P})$, then $\oplus A_n \in L(E', \mathcal{P}')$.

Proof. **a)** and **b)** Take some $d \in \mathbb{N}_0$ and two sets $U, V \subset \mathbb{Z}^{N+1}$ with $\text{dist}(U, V) > d$. Then $P'_V(\oplus A_n)P'_U = \oplus C_n$, where

$$C_n = \begin{cases} P_{V^n} A_n P_{U^n} & \text{if } n \in \mathbb{N}, \\ P_{V^n} P_{U^n} & \text{if } n \notin \mathbb{N}, \end{cases}$$

with $U^n = \{m \in \mathbb{Z}^N : (m, n) \in U\}$, $V^n = \{m \in \mathbb{Z}^N : (m, n) \in V\}$ for $n \in \mathbb{Z}$, and $P_\emptyset = 0$. But from

$$\text{dist}(U^n, V^n) \geq \text{dist}(U, V) > d, \quad n \in \mathbb{Z}$$

and $\|P'_V(\oplus A_n)P'_U\| = \sup \|C_n\|$, we get the claim in **a)** and **b)**, respectively.

c) Take $k, m \in \mathbb{N}$, and note that $P'_k(\oplus A_n)Q'_m = \oplus C_n$ with

$$C_n = \begin{cases} P_k A_n Q_m & \text{if } n \in \mathbb{N} \text{ and } |n| \leq k, \\ 0 & \text{if } n \notin \mathbb{N} \text{ or } |n| > k, \end{cases}$$

whenever $m > k$. Together with condition (6.5) we get that, for all $k \in \mathbb{N}$, $\|P'_k(\oplus A_n)Q'_m\| = \sup \|C_n\| \rightarrow 0$ as $m \rightarrow \infty$. The symmetric property, $\|Q'_m(\oplus A_n)P'_k\| \rightarrow 0$ as $m \rightarrow \infty$, is shown analogously. ■

Stability and Stacked Operators

Our intuition now tells us that the stability of the operator sequence is very closely related with the stacked operator's invertibility at infinity.

Proposition 6.11 Consider a bounded sequence $(A_n)_{n \in \mathbb{N}}$ in $L(E)$, every operator of which is invertible at infinity; that is

$$A_n B_n = I + K_n \quad \text{and} \quad B_n A_n = I + L_n \quad (6.9)$$

for all $n \in \mathbb{N}$, with some $B_n \in L(E)$ and $K_n, L_n \in K(E, \mathcal{P})$, and, in addition, suppose that (B_n) is bounded. Then the stacked operator $\oplus A_n$ is invertible at infinity if and only if (A_n) is stable.

Proof. Suppose there is a $n_* \in \mathbb{N}$ such that the set $\{A_n\}_{n \in \mathbb{N}, n > n_*}$ is uniformly invertible. Then, for every $n \in \mathbb{N}$, put

$$R_n := \begin{cases} A_n^{-1} & \text{if } A_n^{-1} \text{ exists,} \\ B_n & \text{otherwise.} \end{cases}$$

By the uniform boundedness of the sets $\{A_n^{-1}\}_{n \in \mathbb{N}, n > n_*}$ and $\{B_n\}_{n \in \mathbb{N}}$, the operator $\oplus R_n$ is bounded. We show that $K := (\oplus A_n)(\oplus R_n) - I'$ is in $K(E', \mathcal{P})$. Clearly, $K = \oplus K_n$, where

$$K_n = \begin{cases} A_n R_n - I & \text{if } n \in \mathbb{N}, \\ 0 & \text{if } n \notin \mathbb{N}. \end{cases}$$

Take $\varepsilon > 0$ arbitrary, and choose $m \in \mathbb{N}$ such that $m > n_*$ and

$$\sup_{1 \leq n \leq n_*} \left\{ \|Q_m K_n\|, \|K_n Q_m\|, \|Q_m L_n\|, \|L_n Q_m\| \right\} < \varepsilon.$$

Then $Q'_m K = \oplus L_n$, where

$$L_n = \begin{cases} A_n R_n - I & \text{if } n \in \mathbb{N} \text{ and } n > m, \\ Q_m(A_n R_n - I) & \text{if } n \in \mathbb{N} \text{ and } n \leq m, \\ 0 & \text{if } n \notin \mathbb{N}, \end{cases}$$

showing that, by the definition of R_n , all layers L_n with $n > n_*$ are zero, whence $\|Q'_m K\| = \sup_{n \leq n_*} \|Q_m(A_n R_n - I)\| < \varepsilon$. Analogously, we see that also $\|K Q'_m\| < \varepsilon$, showing that $K = (\oplus A_n)(\oplus R_n) - I' \in K(E', \mathcal{P})$. By the same argument one easily checks that also $(\oplus R_n)(\oplus A_n) - I' \in K(E', \mathcal{P})$, and hence that $\oplus A_n$ is invertible at infinity.

Now suppose that $\oplus A_n$ is invertible at infinity. Then, by Lemma 4.5, $\oplus A_n$ is weakly invertible at infinity, which means that there is an $m \in \mathbb{N}$ such that

$$Q'_m(\oplus A_n)B = Q'_m = C(\oplus A_n)Q'_m$$

holds with some $B, C \in L(E')$. If we put $U := \{(k, n) \in \mathbb{Z}^{N+1} : k \in \mathbb{Z}^N, n > m\}$ and multiply the left equality by P'_U from the left and the right equality from the right, we get

$$P'_U(\oplus A_n)B = P'_U = C(\oplus A_n)P'_U$$

since $P'_U Q'_m = P'_U = Q'_m P'_U$. But this equality clearly shows that $\oplus_{n > m} A_n$ is invertible, and hence the set $\{A_n\}_{n > m}$ is uniformly invertible, by Lemma 6.8. ■

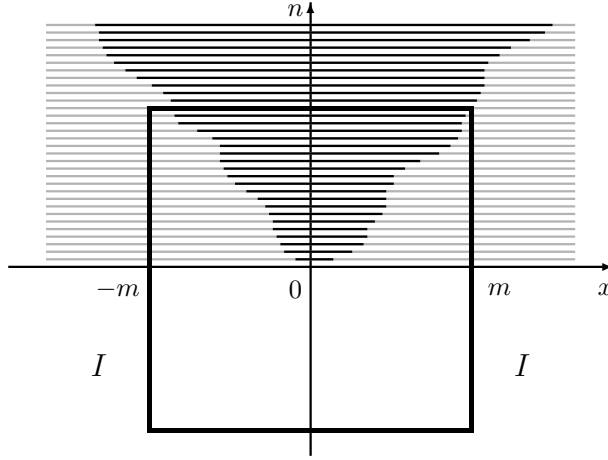


Figure 6: Illustration of the proof of Proposition 6.11 for the finite section method. If (A_n) is stable then everything outside a sufficiently large cube is (locally) invertible.

6.2 The Finite Section Method

As in the previous subsection, let $E = E^p(X) = \ell^p(\mathbb{Z}^N, X)$ with $N \in \mathbb{N}$, $p \in [1, \infty]$ and a Banach space X , and put $E' = \ell^p(\mathbb{Z}^{N+1}, X)$.

If we consider the finite section method (FSM)

$$A_n = A_{[n]} = P_{\Omega_n} A P_{\Omega_n} + Q_{\Omega_n}, \quad n \in \mathbb{N} \quad (6.10)$$

for $A \in L(E)$ with a monotonously increasing sequence (Ω_n) of bounded sets in \mathbb{Z}^N , then the conditions of Proposition 6.11 are clearly satisfied with $B_n = I$.

Proposition 6.12 *If $(A_n) = (A_{[n]})$ is the finite section sequence (6.10) for an operator $A \in L(E)$ then the sequence (A_n) is stable if and only if the stacked operator $\oplus A_n$ is invertible at infinity.*

If we take this result together with Corollary 6.5 we get that the finite section method is applicable to $A \in L(E, \mathcal{P})$ iff A is invertible and $\oplus A_n$ with A_n from (6.10) is invertible at infinity.

The latter can be studied, using the results of Section 5.2, in terms of limit operators of $\oplus A_n$ if moreover A is rich and band-dominated. Note that in this case also $\oplus A_n$ is rich (which can be seen as in the proof of [106, Proposition 4.1]) and band-dominated (by Proposition 6.10) on E' .

We will now follow this idea for the concrete case where the sequence (Ω_n) of truncation geometries is given by homothetic copies of a convex polytope $\Omega \subset \mathbb{R}^N$.

Definition 6.13 Let $v \in \mathbb{N}$ and $\omega_1, \dots, \omega_v \in \mathbb{Z}^N$ be such that 0 is an interior point (w.r.t. \mathbb{R}^N) of the polytope $\Omega := \text{conv}\{\omega_1, \dots, \omega_v\} \subseteq \mathbb{R}^N$. Sets $\Omega \subset \mathbb{R}^N$ that can be written in this form will henceforth be referred to as valid polytopes.

Now, for every $n \in \mathbb{N}$, put

$$\Omega_n := n\Omega \cap \mathbb{Z}^N, \quad \Pi_n := P_{\Omega_n} \quad \text{and} \quad \Theta_n := Q_{\Omega_n}. \quad (6.11)$$

Let $\Gamma := \partial\Omega$ be the boundary of Ω and, for every infinite subset $\mathbb{I} \subset \mathbb{N}$ and every $n \in \mathbb{N}$, put

$$\Gamma_n := n\Gamma \cap \mathbb{Z}^N \quad \text{and then let} \quad \Gamma_{\mathbb{I}} := \bigcup_{n \in \mathbb{I}} \Gamma_n. \quad (6.12)$$

For a sequence $h = (h(1), h(2), \dots) \subseteq \Gamma_{\mathbb{I}}$, say $h(k) \in \Gamma_{m_k}$ for some $m_k \in \mathbb{I}$, and a set $S \subseteq \mathbb{Z}^N$, we call S the *geometric limit* of Ω w.r.t. h and write $S = \Omega_h$ if, for every $m \in \mathbb{N}$, there exists a $k_0 \in \mathbb{N}$ such that

$$(\Omega_{m_k} - h(k)) \cap \{-m, \dots, m\}^N = S \cap \{-m, \dots, m\}^N, \quad k \geq k_0.$$

Remark 6.14 a) Note that in this case $V_{-h(k)}\Pi_{m_k}V_{h(k)}$ is \mathcal{P} -convergent to P_S as $k \rightarrow \infty$. For a polytope Ω , the only candidates for the geometric limit S w.r.t. a sequence $h \subseteq \Gamma_{\mathbb{I}}$ are intersections of finitely many half spaces and \mathbb{Z}^N (discrete half spaces, edges, corners, etc.).

b) Also note that, for a valid polytope Ω , one has $\Gamma_m \cap \Gamma_n = \emptyset$ if $m \neq n$, so that for every $h(k) \in \Gamma_{\mathbb{I}}$ there is exactly one $m_k \in \mathbb{I}$ with $h(k) \in \Gamma_{m_k}$. \square

Given a rich operator $A \in BDO(E)$ and a valid polytope $\Omega \in \mathbb{R}^N$, we put

$$\begin{aligned} &\mathcal{H}_{\Omega}(A) \\ &:= \{ h = (h(1), h(2), \dots) : h(k) \in \Gamma_{\mathbb{N}} \forall k, |h(k)| \rightarrow \infty, A_h \text{ exists, } \Omega_h \text{ exists} \} \end{aligned}$$

and we define the so-called *stability spectrum* of A w.r.t. Ω as

$$\sigma_{\Omega}^{\text{stab}}(A) := \{A\} \cup \{P_{\Omega_h}A_hP_{\Omega_h} + Q_{\Omega_h} : h \in \mathcal{H}_{\Omega}(A)\}. \quad (6.13)$$

Then the following theorem holds.

Theorem 6.15 Under the above conditions, the following are equivalent.

- (i) The sequence $(A_n) = (\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{N}}$ is stable.
- (ii) The operator $\oplus A_n$ is invertible at infinity.
- (iii) All operators in $\sigma_{\Omega}^{\text{stab}}(A)$ are invertible and their inverses are uniformly bounded.

For a proof of this theorem we refer to Theorem 6.17 below of which Theorem 6.15 is a special case with $\mathbb{I} = \mathbb{N}$. Theorem 6.15 generalises results of [142], [143, §6.2], [153] and [106, §4.1] from dimensions $N = 1$ and $N = 2$ to arbitrary $N \in \mathbb{N}$. For $N = 2$ it corrects a small mistake in the literature (see (6.22) and Example 6.22 below).

We will now look at two different strategies for the case when the FSM, as discussed so far, is not applicable or can not be shown to be applicable.

6.3 Strategy 1: Passing to Subsequences

After a look at the following example, where the FSM clearly fails, we will generalise Theorem 6.15 to subsequences of finite sections.

Example 6.16 Let $N = 1$ and consider the operator A induced by the block diagonal matrix

$$\text{diag} \left(\cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \right)$$

with the single 1 entry at position zero. Then $A = A^{-1}$ is invertible and, for $\Omega = [-1, 1]$, its truncations $\Pi_n A \Pi_n$ correspond to the finite $(2n+1) \times (2n+1)$ matrices

$$\text{diag} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

if n is even and to

$$\text{diag} \left(0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

if n is odd. So sequence (6.10) is not stable since all its entries with an odd n are non-invertible. The associated set $\sigma_{\Omega}^{\text{stab}}(A)$ consists in this example of five operators. They are A ,

$$B = \text{diag} \left(\cdots, 1, 1, 1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \right),$$

$$C = \text{diag} \left(\cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1, 1, 1, \cdots \right),$$

$$D = \text{diag} \left(\cdots, 1, 1, 0, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \cdots \right),$$

$$F = \text{diag} \left(\cdots, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0, 1, 1, \cdots \right),$$

out of which only A , B and C are invertible. \square

6.3.1 The Philosophy

As we have just seen, the finite section method cannot be expected to simply work for a given operator A . But in some cases it is possible to “adjust” the method to the operator at hand by choosing the right geometry Ω and an appropriate subsequence of (6.10). The philosophy here is to give the operator A the chance to impose some of its “personality” on the (otherwise too “impersonal”) method of finite sections. In the previous example, for instance, one simply has to remove all elements from the sequence (6.10) that correspond to an odd value of n to get a stable approximation method for A (or alternatively, one could replace $\Omega = [-1, 1]$ by $[-2, 2]$ and work with the whole sequence (6.10)). We believe that, for a given operator A , finding the right geometry Ω and an appropriate index set $\mathbb{I} \subseteq \mathbb{N}$ such that $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is stable is a major task in the numerical analysis of the equation $Au = b$. The following observation (6.14) helps to translate this task into a different, and sometimes more tractable, language.

In [145] the following observation was made in the case ($N = 1$):

An infinite subsequence $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ of (6.10), with index set $\mathbb{I} \subseteq \mathbb{N}$, is stable iff all operators in an associated set $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ are invertible with uniformly bounded inverses.

The set $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ depends, in addition to A and Ω , on the index set $\mathbb{I} \subseteq \mathbb{N}$. It holds that $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A) \subseteq \sigma_{\Omega, \mathcal{J}}^{\text{stab}}(A)$ if $\mathbb{I} \subseteq \mathcal{J} \subseteq \mathbb{N}$ and $\sigma_{\Omega, \mathbb{N}}^{\text{stab}}(A) = \sigma_{\Omega}^{\text{stab}}(A)$.

We call $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ the *stability spectrum of A with respect to Ω and index set $\mathbb{I} \subseteq \mathbb{N}$* .

(6.14)

This generalisation, (6.14), of the $N = 1$ version of Theorem 6.15 from $\mathbb{I} = \mathbb{N}$ to $\mathbb{I} \subseteq \mathbb{N}$ has two important consequences:

Firstly, if the whole sequence (6.10) is not stable then one might be able, via the new result, to detect a stable subsequence (with infinite index set $\mathbb{I} \subseteq \mathbb{N}$, say) and to solve (6.3) for $n \in \mathbb{I}$ only – thereby still approximately solving $Au = b$.

Secondly, the observation (6.14) was used to remove the uniform boundedness condition from the same statement (6.14) and hence also from Theorem 6.15 in the case $N = 1$, which is the main result of [145].

If we put $\mathbb{I} = 2\mathbb{N}$ and $\mathbb{J} = 2\mathbb{N} + 1$ in Example 6.16 then it turns out that $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A) = \{A, B, C\}$ while $\sigma_{\Omega, \mathbb{J}}^{\text{stab}}(A) = \{A, D, F\}$, so that, by (6.14), the finite section subsequence corresponding to all even n is stable. Of course this just confirms what we already observed directly in Example 6.16 but there are, however, examples in which the detection of a less obvious stable subsequence of (6.10) is possible via (6.14).

We will generalise statement (6.14) from dimension $N = 1$ to arbitrary dimensions $N \geq 1$. The question of the uniform boundedness condition in (6.14), however, is much more subtle if $N > 1$ than it was in [145] for $N = 1$. We will say a bit more about this later.

6.3.2 The Stability Theorem for Subsequences of the FSM

Given a rich operator $A \in BDO(E)$, a valid polytope $\Omega \in \mathbb{R}^N$, and an index set $\mathbb{I} = \{n_1, n_2, \dots\} \subseteq \mathbb{N}$ with $n_1 < n_2 < \dots$, we put

$$\begin{aligned} \mathcal{H}_{\Omega, \mathbb{I}}(A) \\ := \{h = (h(1), h(2), \dots) : h(k) \in \Gamma_{\mathbb{I}} \forall k, |h(k)| \rightarrow \infty, A_h \text{ exists, } \Omega_h \text{ exists}\}, \end{aligned}$$

and

$$\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A) := \{A\} \cup \{P_{\Omega_h} A_h P_{\Omega_h} + Q_{\Omega_h} : h \in \mathcal{H}_{\Omega, \mathbb{I}}(A)\}, \quad (6.15)$$

and let, for every $i \in \mathbb{Z}$,

$$A_i := \Pi_{n_i} A \Pi_{n_i} + \Theta_{n_i}, \quad (6.16)$$

where we put $n_i := 0$ for all $i \in \mathbb{Z} \setminus \mathbb{N}$ and $\Omega_0 := \emptyset$ so that $A_i = I$ then. Then the following theorem holds.

Theorem 6.17 *Under the above conditions, the following are equivalent.*

- (i) *The sequence*

$$(A_i)_{i=1}^{\infty} = (\Pi_{n_i} A \Pi_{n_i} + \Theta_{n_i})_{i=1}^{\infty} \quad (6.17)$$
is stable.
- (ii) *The operator $\oplus A_i$, with A_i as in (6.16), is invertible at infinity.*
- (iii) *All operators in $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ are invertible and their inverses are uniformly bounded.*

For dimension $N = 1$, this result coincides with a two-sided version of [145, Theorem 3]. As such it generalizes [142, Theorem 3] (also see [143, Theorem 6.2.2], [106, Theorem 4.2] and [153, Theorem 2.7]) from the full sequence $\mathbb{I} = \mathbb{N}$ to arbitrary infinite subsequences with index set $\mathbb{I} \subseteq \mathbb{N}$.

Proof. We start by reformulating (ii) in terms of limit operators of $\oplus A_i$. By Proposition 6.10 and $A \in BDO(E)$ it follows that $\oplus A_i \in BDO(E')$. Since $\oplus \Pi_{n_i}$ and $\oplus A$ are rich if A is rich, we have that $\oplus A_i = (\oplus \Pi_{n_i})(\oplus A)(\oplus \Pi_{n_i}) + I' - (\oplus \Pi_{n_i})$ is rich. Consequently, Theorem 5.9 (iii) is applicable and shows that $\oplus A_i$ is invertible at infinity iff all its limit operators are invertible and their inverses are uniformly bounded.

The equivalence of (i) and (ii) follows from Proposition 6.12.

(ii) \Rightarrow (iii) : Let $\oplus A_i$ be invertible at infinity. From the discussion at the beginning of the proof we know that all limit operators of $\oplus A_i$ are invertible and that there is a uniform upper bound $C > 0$ on the norms of their inverses. We use this to prove (iii). Firstly, take $h = (h(1), h(2), \dots) \subseteq \mathbb{Z}^{N+1} = \mathbb{Z}^N \times \mathbb{Z}$ with $h(m) = (0, m)$ for $m \in \mathbb{N}$. Then, since 0 is an interior point of Ω , it is easy to see that $(\oplus A_i)_h = \oplus A$. From this and the invertibility of all limit operators of $\oplus A_i$ (with uniform bound C on the inverses) we get invertibility of A and $\|A^{-1}\| = \|(\oplus A)^{-1}\| \leq C$. Secondly, take an arbitrary $\alpha = (\alpha(1), \alpha(2), \dots) \in \mathcal{H}_{\Omega, \mathbb{I}}(A)$ with $\alpha(m) \in \Gamma_{n_{\beta(m)}}$ for some $\beta(m) \in \mathbb{N}$ and put $h(m) = (\alpha(m), \beta(m))$ for every $m \in \mathbb{N}$. Then also the following limit operator of $\oplus A_i$ is invertible and the norm of its inverse is bounded by C :

$$\begin{aligned}
\mathcal{P}'\text{-}\lim V'_{-h(m)}(\oplus A_i)V'_{h(m)} &= \mathcal{P}'\text{-}\lim V'_{(-\alpha(m), -\beta(m))}(\oplus A_i)V'_{(\alpha(m), \beta(m))} \\
&= \mathcal{P}'\text{-}\lim_{m \rightarrow +\infty} \bigoplus_{i \in \mathbb{Z}} (V_{-\alpha(m)} A_{\beta(m)+i} V_{\alpha(m)}) = \bigoplus_i \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\alpha(m)} A_{\beta(m)+i} V_{\alpha(m)} \\
&= \bigoplus_i \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\alpha(m)} (\Pi_{n_{\beta(m)+i}} A \Pi_{n_{\beta(m)+i}} + \Theta_{n_{\beta(m)+i}}) V_{\alpha(m)} \tag{6.18} \\
&= \bigoplus_i \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} \left((V_{-\alpha(m)} \Pi_{n_{\beta(m)+i}} V_{\alpha(m)}) (V_{-\alpha(m)} A V_{\alpha(m)}) (V_{-\alpha(m)} \Pi_{n_{\beta(m)+i}} V_{\alpha(m)}) \right. \\
&\quad \left. + V_{-\alpha(m)} \Theta_{n_{\beta(m)+i}} V_{\alpha(m)} \right)
\end{aligned}$$

So in particular, its $i = 0$ -th layer is invertible with its inverse bounded by C . But this operator is equal to $P_{\Omega_\alpha} A_\alpha P_{\Omega_\alpha} + Q_{\Omega_\alpha}$ by the compatibility of \mathcal{P} -lim with addition and composition and by our assumption $\alpha \in \mathcal{H}_{\Omega, \mathbb{I}}(A)$.

(iii) \Rightarrow (ii) : Suppose all operators in $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ are invertible and their inverses are bounded by a constant $C > 1$. We will show that the same holds for all limit operators of $\oplus A_i$ which implies (ii) by the discussion at the beginning of this proof. So let L be an arbitrary limit operator of $\oplus A_i$, say w.r.t. the sequence $h = (h(1), h(2), \dots) \in \mathbb{Z}^{N+1}$ with $h(m) = (\alpha(m), \beta(m))$ where $\alpha(m) \in \mathbb{Z}^N$ and $\beta(m) \in \mathbb{Z}$ for all $m \in \mathbb{N}$ and $|h(m)| \rightarrow \infty$. To understand the operator L , we will pass to a suitable subsequence of h which, clearly, does not change the limit operator L . By passing to a subsequence of h , it can be arranged that one of the following four cases holds.

Case 1. $\beta(m) \not\rightarrow +\infty$. Then we can choose an infinite subset $M \subseteq \mathbb{N}$ such that $\beta|_M$ either tends to $-\infty$ or is bounded. In either case it is easy to see that $L = I'$ and $\|L^{-1}\| = 1 < C$.

Case 2. $\beta(m) \rightarrow +\infty$ and $\text{dist}(\alpha(m), \Omega_{n_{\beta(m)}}) \rightarrow \infty$. Also then $L = I'$.

Case 3. $\beta(m) \rightarrow +\infty$, $\alpha(m) \in \Omega_{n_{\beta(m)}}$ for all m and $\text{dist}(\alpha(m), \Gamma_{n_{\beta(m)}}) \rightarrow \infty$. Note that under these conditions (also see (6.19) and Remark 6.18 below),

$\text{dist}(\alpha(m), n_{\beta(m)}\Gamma) \rightarrow \infty$ as $m \rightarrow \infty$, whence $\mathcal{P}\text{-}\lim V_{-\alpha(m)}\Pi_m V_{\alpha(m)} = I$. Now consider these two subcases:

Case 3.1. If $|\alpha(m)| \rightarrow \infty$, choose an infinite subset M of \mathbb{N} such that A has a limit operator w.r.t. the remaining subsequence $\alpha|_M$, for simplicity again denoted by α , which is possible since A is rich. Then $L = \oplus_i A_\alpha$, which is invertible since A_α is invertible by the invertibility of A . Also $\|L^{-1}\| = \|A_\alpha^{-1}\| \leq \|A^{-1}\| \leq C$.

Case 3.2. If $|\alpha(m)| \not\rightarrow \infty$ then α has a bounded and therefore even a constant subsequence. So take $M \subseteq \mathbb{N}$ such that $\alpha|_M \equiv: d \in \mathbb{Z}^N$. Then $L = \oplus_i (V_{-d} A V_d)$ is invertible since A is invertible, and $\|L^{-1}\| = \|A^{-1}\| \leq C$.

Case 4. $\beta(m) \rightarrow +\infty$ and $\text{dist}(\alpha(m), \Gamma_{n_{\beta(m)}})$ remains bounded. For $m \in \mathbb{N}$ and $i \in \mathbb{Z}$, put

$$\begin{aligned} \gamma^{(i)}(m) &:= \operatorname{argmin}\{|\alpha(m) - \gamma| : \gamma \in \Gamma_{n_{\beta(m)+i}}\} \\ \text{and} \quad \delta^{(i)}(m) &:= \alpha(m) - \gamma^{(i)}(m). \end{aligned}$$

By our condition, $\delta^{(0)}$ is bounded in \mathbb{Z}^N . Choose $M \subseteq \mathbb{N}$ such that $\delta^{(0)}|_M$ is constant, say equal to $\delta_\infty^{(0)} \in \mathbb{Z}^N$. For every $i \in \mathbb{Z} \setminus \{0\}$, $\delta^{(i)}|_M$ either tends to infinity (in absolute value) or it has a constant subsequence. By a simple diagonal construction, we can pass to a subset of M (for simplicity again denoted by M) such that, for every $i \in \mathbb{Z}$, $\delta^{(i)}|_M$ either tends to infinity or is constant, say equal to $\delta_\infty^{(i)} \in \mathbb{Z}^N$. The set of all $i \in \mathbb{Z}$ for which $\delta^{(i)}|_M$ is constant will be denoted by $\mathbb{Z}_{\text{finite}}$; otherwise, i.e. if $|\delta^{(i)}|_M| \rightarrow \infty$, we will write $i \in \mathbb{Z}_\infty^+$ if $\alpha(m) \in \Omega_{n_{\beta(m)+i}}$ as $m \rightarrow \infty$ and $i \in \mathbb{Z}_\infty^-$ if $\alpha(m) \notin \Omega_{n_{\beta(m)+i}}$ as $m \rightarrow \infty$ (note that it can be arranged in the choice of the subsequence above that $\alpha(m)$ is either \in or \notin of $\Omega_{n_{\beta(m)+i}}$ for all $m > m_0$, say). Finally, again by a diagonal procedure, pass to an infinite subset of M , again denoted by M , such that, for every $i \in \mathbb{Z}$, the geometric limit $\Omega_{\gamma^{(i)}|_M}$ exists (see the construction in the proof of [140, Proposition 5] or [143, Theorem 2.1.16]) and the limit operators $A_{\gamma^{(i)}|_M}$ and $A_{\alpha|_M}$ exist (possible since A is rich). Abbreviating $\alpha|_M$, $\gamma^{(i)}|_M$ and $\delta^{(i)}|_M$ by α , $\gamma^{(i)}$ and $\delta^{(i)}$, respectively, and repeating the previously performed computations up to line (6.18), we get that $L = \oplus L_i$, where the i -th layer of L is

$$\begin{aligned} L_i &= \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\alpha(m)} (\Pi_{n_{\beta(m)+i}} A \Pi_{n_{\beta(m)+i}} + \Theta_{n_{\beta(m)+i}}) V_{\alpha(m)} \\ &= \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\delta^{(i)}(m)} V_{-\gamma^{(i)}(m)} (\Pi_{n_{\beta(m)+i}} A \Pi_{n_{\beta(m)+i}} + \Theta_{n_{\beta(m)+i}}) V_{\gamma^{(i)}(m)} V_{\delta^{(i)}(m)} \\ &= \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\delta_\infty^{(i)}} \left(P_{\Omega_{n_{\beta(m)+i}-\gamma^{(i)}(m)}} V_{-\gamma^{(i)}(m)} A V_{\gamma^{(i)}(m)} P_{\Omega_{n_{\beta(m)+i}-\gamma^{(i)}(m)}} \right. \\ &\quad \left. + Q_{\Omega_{n_{\beta(m)+i}-\gamma^{(i)}(m)}} \right) V_{\delta_\infty^{(i)}} \\ &= V_{-\delta_\infty^{(i)}} \left(P_{\Omega_{\gamma^{(i)}}} A_{\gamma^{(i)}} P_{\Omega_{\gamma^{(i)}}} + Q_{\Omega_{\gamma^{(i)}}} \right) V_{\delta_\infty^{(i)}} \end{aligned}$$

if $i \in \mathbb{Z}_{\text{finite}}$ and

$$\begin{aligned} L_i &= \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} V_{-\alpha(m)} (\Pi_{n_{\beta(m)+i}} A \Pi_{n_{\beta(m)+i}} + \Theta_{n_{\beta(m)+i}}) V_{\alpha(m)} \\ &= \mathcal{P}\text{-}\lim_{m \rightarrow +\infty} \left(P_{\Omega_{n_{\beta(m)+i}-\alpha(m)}} V_{-\alpha(m)} A V_{\alpha(m)} P_{\Omega_{n_{\beta(m)+i}-\alpha(m)}} + Q_{\Omega_{n_{\beta(m)+i}-\alpha(m)}} \right) \\ &= \begin{cases} A_\alpha, & i \in \mathbb{Z}_\infty^+, \\ I, & i \in \mathbb{Z}_\infty^- \end{cases} \end{aligned}$$

if $i \in \mathbb{Z} \setminus \mathbb{Z}_{\text{finite}}$. In either case, L_i is invertible and $\|L_i^{-1}\| \leq C$ by (iii). Consequently, L is invertible and $\|L^{-1}\| = \sup_i \|L_i^{-1}\| \leq C$. ■

Remark 6.18 In case 3 of the proof we used the implication

$$\text{dist}(x_n, \Gamma_n) \rightarrow \infty \implies \text{dist}(x_n, n\Gamma) \rightarrow \infty \quad (6.19)$$

for arbitrary points $x_n \in \mathbb{Z}^N$. It is easy to see that (6.19) is equivalent to

$$\text{dist}(x_n, \Gamma_n) \leq \text{dist}(x_n, n\Gamma) + \delta \quad (6.20)$$

with a global finite constant $\delta := \max_{n \in \mathbb{N}} \max_{\gamma \in n\Gamma} \text{dist}(\gamma, \Gamma_n)$, so that the δ -neighbourhood of every $\gamma \in n\Gamma$ contains a point from Γ_n (that is, Γ_n is “relatively dense” in $n\Gamma$) for every $n \in \mathbb{N}$. For convex polytopes Ω with vertices in \mathbb{Z}^N , it is clear that (6.20) and hence (6.19) holds – with a constant δ that is the maximum of the respective constants for the finitely many facets of Ω . □

6.3.3 Starlike Sets Instead of Convex Polytopes

Definition 6.19 We say that $\Omega \subset \mathbb{R}^N$ is a valid starlike set if $\Omega \neq \emptyset$ is bounded and if, for all $x \in \Omega$ and $\alpha \in [0, 1)$, the point αx is an interior point of Ω .

So in particular, 0 is an interior point of every valid starlike set. Moreover, all bounded convex sets $\Omega \subset \mathbb{R}^N$ with interior point 0 are valid starlike sets. We claim that we can prove a version of Theorem 6.17 in the much more general setting of a valid starlike set Ω . Our reason for this choice of geometry (as opposed to convex polytopes with integer vertices) is, of course, more generality (including e.g. the ball $\Omega = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1^2 + \dots + x_N^2 \leq R\}$) but at the same time still to make sure that the boundaries of $m\Omega$ and $n\Omega$ are disjoint if $m \neq n$ (recall Remark 6.14.b and the definition of the geometric limit Ω_h). However, for valid starlike sets, the implication (6.19) is in general not true:

Example 6.20 For example, let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^3 + |x_2|^3 \leq 1\}$ and put $\Gamma := \partial\Omega$ and $\Gamma_n := (n\Gamma) \cap \mathbb{Z}^N$ for $n = 1, 2, \dots$. Then it is well-known (this is a result of Euler and of course a special case of Fermat’s Last Theorem) that every $\Gamma_n = \{(x_1, x_2) \in \mathbb{Z}^2 : |x_1|^3 + |x_2|^3 = n^3\}$ only consists of the four points $(\pm n, 0)$ and $(0, \pm n)$, so that (6.19) and (6.20) clearly fail. □

The workaround is to use “fat boundaries”: If we replace the first definition in (6.12) by

$$\Gamma_n := (n\Gamma + F) \cap \mathbb{Z}^N \quad \text{with} \quad F = (-1/2, 1/2]^N \quad (6.21)$$

then (6.19) and (6.20) always hold with $\delta = 1$ (distances measured in the $|\cdot|_\infty$ metric). As the rest of the proof of Theorem 6.17 carries over to valid starlike sets, we get the following generalisation.

Theorem 6.21 *Let $A \in BDO(E)$ be rich, $\Omega \subset \mathbb{R}^N$ a valid starlike set, and $\mathbb{I} = \{n_1, n_2, \dots\} \subseteq \mathbb{N}$ an infinite index set with $n_1 < n_2 < \dots$. Then, with $\Gamma_{\mathbb{I}}$, $\mathcal{H}_{\Omega, \mathbb{I}}(A)$ and $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ defined as before but now with Γ_n given by (6.21) for every $n \in \mathbb{N}$, the following are equivalent.*

- (i) *The sequence (6.17) is stable.*
- (ii) *The operator $\oplus A_i$, with A_i as in (6.16), is invertible at infinity.*
- (iii) *All operators in $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ are invertible and their inverses are uniformly bounded.*

Unlike for the polytopes in Theorem 6.17, for valid starlike sets $\Omega \subset \mathbb{R}^N$, there can be an infinite amount of different geometric limits Ω_h . For example, if $N = 2$ and Ω is the unit disk $\{(x_1, x_2) : |x_1|^2 + |x_2|^2 \leq 1\}$ in \mathbb{R}^2 , then all discrete half planes $\{(x_1, x_2) \in \mathbb{Z}^2 : ax_1 + bx_2 < 0\}$ with $(a, b) \in \mathbb{R}^2 \setminus \{0\}$ occur as geometric limits, but also the same sets with one additional point (x_1, x_2) with $ax_1 + bx_2 = 0$ (e.g. look at $h(n) = (c, n)$ for fixed $c \in \mathbb{Z}$ and all $n \in \mathbb{N}$ – note that $h(n) \in \Gamma_n$ for all sufficiently large n – to see this effect for the case $(a, b) = (0, 1)$) are geometric limits of the disk Ω . In Example 6.20 the same discrete half planes occur but only those with a fully vertical or horizontal ascent appear again with an additional point.

6.3.4 Examples

As a particularly illustrative and not too difficult class of examples, we will look at operators that are induced by an adjacency matrix. Therefore, let \mathcal{E} denote a set of pairwise disjoint doubletons $\{i, j\}$ with $i, j \in \mathbb{Z}^N$, $i \neq j$, and put

$$a_{ij} := \begin{cases} I_X, & \text{if } \{i, j\} \in \mathcal{E} \text{ or } i = j \notin \bigcup_{e \in \mathcal{E}} e, \\ 0_X, & \text{otherwise,} \end{cases}$$

for all $i, j \in \mathbb{Z}^N$, where I_X and 0_X stand for the identity and zero operator, respectively, on the Banach space X at hand. Then $(a_{ij})_{i, j \in \mathbb{Z}^N}$ is the extended

adjacency matrix of the undirected graph $\mathcal{G} = (\mathbb{Z}^N, \mathcal{E})$ with vertex set \mathbb{Z}^N and edges \mathcal{E} . We write $Adj(\mathcal{G})$ for the operator that is induced by this matrix (a_{ij}) and note that $Adj(\mathcal{G})$ is band-dominated iff $b := \sup_{\{i,j\} \in \mathcal{E}} |i - j|$ is finite, in which case $Adj(\mathcal{G})$ is even a band operator with band-width b .

If applied to an element $u \in E = \ell^p(\mathbb{Z}^N, X)$, the operator $Adj(\mathcal{G})$ “swaps” the values $u(i)$ and $u(j)$ around if $\{i, j\}$ is an edge of \mathcal{G} , and it leaves all values $u(k)$ untouched for which $k \in \mathbb{Z}^N$ is not part of an edge of \mathcal{G} . From this it is obvious that $\|Adj(\mathcal{G})\| = 1$ and that $Adj(\mathcal{G})$ is invertible and coincides with its inverse. Moreover, it is clear that, for $n \in \mathbb{N}$, the n -th finite section $\Pi_n Adj(\mathcal{G}) \Pi_n + \Theta_n$ is invertible iff each edge $e \in \mathcal{E}$ has either both or no vertices in $\Omega_n = n\Omega \cap \mathbb{Z}^N$. In the latter case, $\Pi_n Adj(\mathcal{G}) \Pi_n + \Theta_n$ equals $Adj(\mathcal{G}_n)$, where $\mathcal{G}_n = (\mathbb{Z}^N, \mathcal{E} \cap \Omega_n^2)$, is again its own inverse and has norm 1. So we get that, for $A = Adj(\mathcal{G})$, the sequence (6.17) is stable iff, for all sufficiently large $n \in \mathbb{I}$, each edge $e \in \mathcal{E}$ has either both or no vertices in Ω_n .

Note that Example 6.16 was already of the form $A = Adj(\mathcal{G})$, namely with $N = 1$ and

$$\mathcal{E} = \left\{ \dots, \{-4, -3\}, \{-2, -1\}, \{1, 2\}, \{3, 4\}, \dots \right\}.$$

Here Ω_n separates the vertices of $\{-n-1, -n\}$ and also $\{n, n+1\}$ if n is odd.

We continue with two examples demonstrating that two particular sets of operators that are closely related to $\sigma_\Omega^{\text{stab}}(A)$ are actually not stability spectra (meaning that Theorem 6.15 is incorrect with $\sigma_\Omega^{\text{stab}}(A)$ replaced by any of them) if $N > 1$. These two “non-replacements” for $\sigma_\Omega^{\text{stab}}(A)$ are

$$\{A\} \cup \bigcup_{x \in \Gamma} \{P_{\Omega_x} B P_{\Omega_x} + Q_{\Omega_x} : B \in \sigma_x^{\text{op}}(A)\} \quad (6.22)$$

and

$$\{A\} \cup \bigcup_{x \in \Gamma} \{P_{\Omega_x} B P_{\Omega_x} + Q_{\Omega_x} : B \in \sigma_{x, \text{ray}}^{\text{op}}(A)\}, \quad (6.23)$$

where $\Gamma = \partial\Omega$ and, for every $x \in \Gamma$, $\Omega_x \subseteq \mathbb{Z}^N$ is the limit of $n(\Omega - x) \cap \mathbb{Z}^N$ as $n \rightarrow \infty$ in the sense that, for each $m \in \mathbb{N}$,

$$n(\Omega - x) \cap \{-m, \dots, m\}^N = \Omega_x \cap \{-m, \dots, m\}^N$$

for all sufficiently large $n \in \mathbb{N}$. Finally, $\sigma_x^{\text{op}}(A)$ is the set of all limit operators A_h of A with respect to sequences $h = (h(1), h(2), \dots) \subseteq \mathbb{Z}^N$ going to infinity in the direction x , i.e. $h(n)/|h(n)| \rightarrow x/|x|$, and $\sigma_{x, \text{ray}}^{\text{op}}$ is the set of all limit operators A_h with respect to sequences of the form $h = ([m_1 x], [m_2 x], \dots) \subseteq \mathbb{Z}^N$ where (m_n) is an unbounded monotonously increasing sequence of positive reals and, as introduced at the very beginning, $[\cdot]$ means componentwise rounding to the nearest integer.

Example 6.22 Take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = \text{Adj}(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$ and

$$\mathcal{E} = \left\{ \{ (k^2 - k - 1, k^2), (k^2 - k, k^2) \} : k = 1, 2, \dots \right\}.$$

Then, with respect to $h = (h(1), h(2), \dots)$ with $h(k) = (k^2 - k - 1, k^2) \in \mathbb{Z}^2$, the limit operator of A exists and is equal to $B = \text{Adj}(\mathcal{G}')$, where $\mathcal{G}' = (\mathbb{Z}^2, \left\{ \{ (0, 0), (1, 0) \} \right\})$. Since $h(k)/|h(k)| \rightarrow x/|x|$ with $x = (1, 1)$, we have that $B \in \sigma_x^{\text{op}}(A)$. But $\Omega_x = \{\dots, -1, 0\}^2$ separates $(0, 0)$ from $(1, 0)$ so that $P_{\Omega_x} B P_{\Omega_x} + Q_{\Omega_x} \in (6.22)$ is not invertible. However, the whole finite section sequence (6.10) is stable since all edges $e \in \mathcal{E}$ have either both or no points in Ω_n , so that $\Pi_n A \Pi_n + \Theta_n = \text{Adj}(\mathcal{G}_n)$ with $\mathcal{G}_n = (\mathbb{Z}^2, \mathcal{E} \cap \Omega_n^2)$ for every $n \in \mathbb{N}$. So (6.22) is not a valid replacement of (6.15) as stability spectrum.

Note that the element of (6.15) that corresponds to the limit operator $B = A_h$ of A is $P_{\Omega_h} B P_{\Omega_h} + Q_{\Omega_h}$ with $\Omega_h = \mathbb{Z} \times \{\dots, -1, 0\}$ instead of $\{\dots, -1, 0\}^2$, which is again equal to B (since both $(0, 0)$ and $(1, 0)$ are in Ω_h) and hence invertible. \square

Similarly, we can rule out (6.23) as stability spectrum by the following example:

Example 6.23 Again take $N = 2$, $\Omega = [-1, 1]^2$ and let $A = \text{Adj}(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$ and

$$\mathcal{E} = \left\{ \{ (k^2 - k, k^2), (k^2 - k, k^2 + 1) \} : k = 1, 2, \dots \right\}.$$

Then, with respect to $h = (h(k))_{k \in \mathbb{N}}$ with $h(k) = (k^2 - k, k^2) \in \mathbb{Z}^2$, the limit operator of A exists and is equal to $B = \text{Adj}(\mathcal{G}')$, where $\mathcal{G}' = (\mathbb{Z}^N, \left\{ \{ (0, 0), (0, 1) \} \right\})$.

Again $B \in \sigma_x^{\text{op}}(A)$ with $x = (1, 1)$. But $B \notin \sigma_{x, \text{ray}}^{\text{op}}(A)$ neither is B in $\sigma_{y, \text{ray}}^{\text{op}}(A)$ for any other $y \in \Gamma$! In fact, it holds that $\sigma_{y, \text{ray}}^{\text{op}}(A) = \{I\}$ for all $y \in \Gamma$, whence (6.23) is elementwise invertible with uniformly bounded inverses. However, the finite section sequence (6.10) is not stable since Ω_n separates $(k^2 - k, k^2)$ from $(k^2 - k, k^2 + 1)$ if $n = k^2$. So also (6.23) is not a valid replacement of (6.15) as stability spectrum.

Note that, for $\mathbb{I} = \mathbb{N}$, (6.15) contains the operator $P_{\Omega_h} B P_{\Omega_h} + Q_{\Omega_h}$ with $\Omega_h = \mathbb{Z} \times \{\dots, -1, 0\}$, which is non-invertible since Ω_h separates $(0, 0)$ from $(0, 1)$. This operator is however removed from (6.15) if we remove all (sufficiently large) square numbers from \mathbb{I} , which matches our direct observation that $\Pi_n A \Pi_n + \Theta_n$ is non-invertible iff n is a square number. \square

It is clear that Examples 6.22 and 6.23 can easily be heaved to dimensions $N > 2$. Let us look at another example, for simplicity also in dimension $N = 2$.

Example 6.24 We look at $A = \text{Adj}(\mathcal{G})$ for $\mathcal{G} = (\mathbb{Z}^2, \mathcal{E})$, where

$$\mathcal{E} = \left\{ \{ (k, 1), (k+1, 0) \} : k = 1, 2, \dots \right\}.$$

It is not hard to see that every limit operator of A is either the identity operator I or the operator $B = \text{Adj}(\mathcal{G}')$ for $\mathcal{G}' = (\mathbb{Z}^2, \mathcal{E}')$, where

$$\mathcal{E}' = \left\{ \{ (k, 1), (k+1, 0) \} : k \in \mathbb{Z} \right\},$$

or it is a translate of B . Looking at B and noting that $B = A_h$ for all sequences $h = (h(1), h(2), \dots)$ with $h(k) = (m_k, 0)$ and $m_k \rightarrow +\infty$, we can say how Ω has to look locally at the intersection z of its boundary Γ with the positive x -axis in order for the finite section method to be stable. Here the upward tangent of Γ at z has to enclose an angle $\alpha \in (90^\circ, 135^\circ]$ with the positively directed x -axis.

So, for example, the finite section sequence of A is stable if Ω is the square $\text{conv}\{(1, 0), (0, 1), (-1, 0), (0, -1)\}$ or the triangle $\text{conv}\{(0, 2), (2, -2), (-2, -2)\}$, whereas it does not even have a stable subsequence if Ω is the square $[-1, 1]^2$. \square

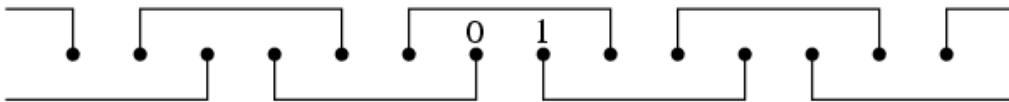
The next example is closely related to Example 6.16.

Example 6.25 a) Let $A = \text{Adj}(\mathcal{G})$ where $\mathcal{G} = (\mathbb{Z}, \mathcal{E})$ is the following infinite graph:



Then, no matter how we choose $\Omega = [a, b]$ with integers $a < 0 < b$, the finite section method does not even have a stable subsequence. A workaround would be to take $\Omega = [-1, 1)$ (which is not a valid polytope in our sense but a valid starlike set) or to increase the dimension to $N = 2$, where we place the edges \mathcal{E} along the x -axis and put $\Omega = \text{conv}\{(-1, 0), (1, 1), (0, -1)\}$, for example. In the latter case, the finite section subsequence corresponding to $\mathbb{I} = 4\mathbb{N} + 1$ turns out to be stable.

b) In contrast to a), there is no workaround whatsoever if $A = \text{Adj}(\mathcal{G})$ with the following graph \mathcal{G} (embedded in dimension $N = 1$ or higher):



For every valid polytope or starlike set Ω and every $n \in \mathbb{N}$, the set Ω_n separates the endpoints of at least two edges of \mathcal{G} so that $\Pi_n A \Pi_n + \Theta_n$ is non-invertible. \square

For any dimension $N \in \mathbb{N}$, any valid set $\Omega \in \mathbb{R}^N$ and any given sequence

$n_1 < n_2 < \dots$ of naturals, one can construct a graph \mathcal{G} in the style¹ of Example 6.25 b) such that $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is stable iff \mathbb{I} is a subset of $\{n_1, n_2, \dots\}$.

6.3.5 Some Specialities in the Case $N = 1$

In this section we let $N = 1$. Not surprisingly, our results are most complete in this case, where we can sharpen and extend much of what was said previously. This is clearly due to the simple geometry of this setting: Firstly, to infinity there are only two ways to go: right or left, and secondly, all valid starlike sets Ω are intervals (open, closed or semi-open) from a to b with reals $a < 0 < b$ (for valid polytopes, the interval is closed and a, b are integers) so that there are only two possibilities for the set Ω_h in (6.15), namely $N_0 = \{0, 1, \dots\}$ and $-N_0 = \{\dots, -1, 0\}$.

The first result is from [145]. We include it here for completeness and because it highlights an important benefit from extending Theorem 6.15 to subsequences.

Proposition 6.26 *If $N = 1$, $\Omega \subset \mathbb{R}$ is a valid polytope, $A \in BDO(E)$ is rich and $\mathbb{I} = \{n_1, n_2, \dots\} \subseteq \mathbb{N}$ is an infinite index set with $n_1 < n_2 < \dots$ then the FSM (6.17) is stable iff A itself and all operators*

$$P_{N_0} C P_{N_0} + Q_{N_0} \quad \text{and} \quad P_{-N_0} B P_{-N_0} + Q_{-N_0}$$

with $C = A_h$ for $h \in \mathcal{H}_{\Omega, \mathbb{I}}(A)$ going to $-\infty$ and $B = A_h$ for $h \in \mathcal{H}_{\Omega, \mathbb{I}}(A)$ going to $+\infty$ are invertible.

So, in particular, the uniform boundedness condition is redundant in all cases $I \subset \mathbb{N}$, including $\mathbb{I} = \mathbb{N}$.

We give the proof later in Section 6.3.6 as a special case of Lemma 6.29, where we discuss possible extensions to $N \geq 2$. Next we show that, in dimension $N = 1$, if the full finite section sequence (6.10) is stable for one valid polytope Ω (i.e. interval $[a, b]$ with integers $a < 0 < b$) then (6.10) has a stable subsequence for all valid polytopes Ω . So conversely, if there exists a valid polytope Ω for which (6.10) has no stable subsequence then there is no valid polytope Ω for which the whole sequence (6.10) is stable.

¹The idea is to take the graph from Example 6.25 b) and to place “gaps” between $a_i := \lceil an_i \rceil$ and $a_i - 1$ and between $b_i := \lfloor bn_i \rfloor$ and $b_i + 1$ for $i = 1, 2, \dots$, where $a < 0$ and $b > 0$ are the unique intersection points of $\Gamma = \partial\Omega$ with the x -axis and $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ stand for rounding up and down to the nearest integer, respectively.

Proposition 6.27 *Let $E = \ell^p(\mathbb{Z}, X)$ with $p \in [1, \infty]$ and a Banach space X , let $A \in BDO(E)$ be a rich operator, and take integers $a < 0 < b$. If the full finite section method $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{N}}$ is stable for $\Omega = [a, b]$ then, for all integers $a' < 0 < b'$, there exists an infinite index set $\mathbb{I} \subseteq \mathbb{N}$ such that the finite section subsequence $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is stable for $\Omega = [a', b']$.*

Proof. Let (6.10) be stable for $\Omega = \Omega^{(1)} = [a, b]$. Putting $\Omega^{(2)} := [a', b']$ for two arbitrary integers $a' < 0 < b'$, $\Gamma^{(i)} := \partial\Omega^{(i)}$ for $i = 1, 2$, and $\mathbb{I} := \{-ab, -2ab, -3ab, \dots\} \subseteq \mathbb{N}$, it is easy to see that

$$\begin{aligned} \Gamma_{\mathbb{I}}^{(2)} &= \bigcup_{n \in \mathbb{I}} n\Gamma^{(2)} = \{na', nb' : n \in \mathbb{I}\} = \{-maba', -mabb' : m \in \mathbb{N}\} \\ &\subseteq \{nab, -nab : n \in \mathbb{N}\} \subseteq \{ka, kb : k \in \mathbb{N}\} = \bigcup_{k \in \mathbb{N}} k\Gamma^{(1)} = \Gamma_{\mathbb{N}}^{(1)} \end{aligned}$$

and consequently $\mathcal{H}_{\Omega^{(2)}, \mathbb{I}}(A) \subseteq \mathcal{H}_{\Omega^{(1)}, \mathbb{N}}(A)$. Since, moreover, for both choices of Ω and all sequences $h = (h(1), h(2), \dots)$ with values in, respectively, $\Gamma_{\mathbb{N}}^{(1)}$ or $\Gamma_{\mathbb{I}}^{(2)}$, it holds that

$$\Omega_h = \begin{cases} \{0, 1, 2, \dots\} & \text{if } h(k) \rightarrow -\infty, \\ \{\dots, -2, -1, 0\} & \text{if } h(k) \rightarrow +\infty, \end{cases}$$

we get that $\sigma_{\Omega^{(2)}, \mathbb{I}}^{\text{stab}}(A) \subseteq \sigma_{\Omega^{(1)}, \mathbb{N}}^{\text{stab}}(A)$. Using Theorem 6.17 (twice), we get that $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is stable for $\Omega = \Omega^{(2)}$. ■

Note that it is not true that if (6.10) has a stable **sub**sequence for one valid polytope Ω then (6.10) has a stable subsequence for all valid polytopes Ω . For example, (6.17) is stable for $A = \text{Adj}(\mathcal{G})$ with $\mathcal{G} = (\mathbb{Z}, \mathcal{E})$ and

$$\mathcal{E} = \left\{ \{-2(2k+1), -2(2k-1)-1\}, \{3(2k-1)+1, 3(2k+1)\} : k \in \mathbb{N} \right\}$$

if one takes $\Omega = [-2, 3]$ and $\mathbb{I} = 2\mathbb{N} + 1$ but there is no stable subsequence of (6.10) for $\Omega = [-1, 1]$.

Example 6.25 b) has shown that, for some operators, the finite section method cannot be “adjusted” via choosing Ω and \mathbb{I} to become stable. We now give a necessary criterion for the existence of an index set $\mathbb{I} \subseteq \mathbb{N}$ such that (6.17) is stable.

Recall the notations from Section 5.4: We abbreviate $P_{\mathbb{N}_0} =: P$ and $I - P =: Q$, and we denote by $\text{ind}_+(A)$ the Fredholm index of $PAP + Q$ and by $\text{ind}_-(A)$ the Fredholm index of $QAQ + P$

Proposition 6.28 *Let $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$ and $A \in BDO(E)$. For the existence of a valid starlike set Ω and an infinite index set $\mathbb{I} \subseteq \mathbb{N}$ such that the*

sequence $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is stable it is necessary, but not sufficient, that A is invertible and $\text{ind}_+(A) = 0$.

Proof. That invertibility of A and $\text{ind}_+(A) = 0$ are not enough for the existence of an Ω and an index set $\mathbb{I} \subseteq \mathbb{N}$ such that (6.17) is stable can be seen in Example 6.25 b) (note that $\text{ind}_+(A) = 0$ there since the adjacency matrix of an undirected graph \mathcal{G} , and hence also $P \text{Adj}(\mathcal{G}) P + Q$, is symmetric).

Now suppose Ω is a valid starlike set (i.e. an interval from $a < 0$ to $b > 0$) and an index set $\mathbb{I} \subseteq \mathbb{N}$ is found such that (6.17) is stable. Then, by Theorem 6.21, we have that A is invertible and $P_{-\mathbb{N}_0} A_h P_{-\mathbb{N}_0} + Q_{-\mathbb{N}_0}$ is invertible for all $h \in \mathcal{H}_{\Omega, \mathbb{I}}(A)$ tending to $+\infty$ (note that A is automatically rich if $X = \mathbb{C}$, see Lemma 5.1). The latter clearly implies that $P_{-\mathbb{N}_0} B P_{-\mathbb{N}_0} + Q_{-\mathbb{N}_0}$ is invertible for some operator $B \in \sigma_+^{\text{op}}(A)$, whence, as a little thought shows,

$$\text{ind}_-(B) = \text{ind}(QBQ + P) = \text{ind}(P_{-\mathbb{N}_0} B P_{-\mathbb{N}_0} + Q_{-\mathbb{N}_0}) = 0.$$

Since A is Fredholm (even invertible), all its limit operators (including B) are invertible, so that also $\text{ind}_+(B) = 0$ holds since $\text{ind}_+(B) + \text{ind}_-(B) = \text{ind}(B) = 0$. By Proposition 5.27 we get that not only B but all operators in $\sigma_+^{\text{op}}(A)$ have plus-index zero, and even more: $\text{ind}_+(A) = 0$. (Analogously, all operators in $\sigma_-^{\text{op}}(A)$ and A itself have minus-index zero, but the latter also follows from $\text{ind}_+(A) = 0$ and $\text{ind}(A) = 0$). ■

In [111] (see Section 7.2.1 below) it was shown that, under the additional condition that A is slowly oscillating, invertibility of A and $\text{ind}_+(A) = 0$ are even sufficient for the stability of the full finite section sequence (6.10) for all valid Ω .

By Proposition 6.28, for an invertible operator A with $\kappa := \text{ind}_+(A) \neq 0$, there is no valid Ω and no index set $\mathbb{I} \subseteq \mathbb{N}$ for which (6.17) is stable. This problem of a nonzero plus-index κ can be simply overcome as follows: Instead of solving $Au = b$, one looks at $V_\kappa Au = V_\kappa b$. Since V_κ is invertible, these two equations are equivalent. Moreover, we have that also $A' := V_\kappa A$ is invertible and

$$\text{ind}_+(A') = \text{ind}_+(V_\kappa A) = \text{ind}_+(V_\kappa) + \text{ind}_+(A) = -\kappa + \kappa = 0.$$

This preconditioning-type procedure of shifting all matrix entries (incl. the right hand side b) down by κ rows is reminiscent of Gohberg's statement that, in a two-sided infinite matrix, "it is every diagonal's right to claim to be the main one" (see page 51 in [81] and the discussion there). Our computations show that, from the perspective of the finite section method, there is one diagonal that deserves being the main diagonal a bit more than the others.

6.3.6 On the Uniform Boundedness Condition in (iii)

Of course, it would be desirable to remove the condition on the uniform boundedness of the inverses from statement (iii) of our Theorems 6.17 and 6.21 for arbitrary dimensions N . For $N \geq 2$ this is a much more delicate problem than for $N = 1$.

What clearly can be said by looking at the proof of Theorem 6.17 is that the uniform boundedness condition (UBC) can be removed from statement (iii) in all cases where it can be removed from Theorem 5.9 (iii). This is known to be the case if one of the following holds

- $p \in \{1, \infty\}$ (see Theorem 5.9 (iv)),
- A is slowly oscillating (see Proposition 5.48),
- A is contained in the Wiener algebra \mathcal{W} (see Theorem 5.19).

For a more general removal of the UBC in dimension $N \geq 2$, we try to generalise the approach that has worked successfully for $N = 1$ in [145]. Therefore, given a rich operator $A \in BDO(E)$ and a valid starlike set $\Omega \subset \mathbb{R}^N$, we will call the infinite index set $\mathbb{I} \subseteq \mathbb{N}$ *sufficient w.r.t. A and Ω* , and write $\mathbb{I} \in \text{suff}(A, \Omega)$, if $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ is either uniformly invertible or not elementwise invertible, i.e. it holds that elementwise invertibility of $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ implies its uniform invertibility.

In what follows, when we use the letters \mathbb{I}, \mathbb{J} and \mathbb{K} for subsets of \mathbb{N} , we always mean infinite subsets. In Example 6.16, one has that for every $\mathbb{I} \subset \mathbb{N}$ and every valid Ω , the set $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ is finite so that, clearly, $\mathbb{I} \in \text{suff}(A, \Omega)$. In general, the following lemma (which is reminiscent of the basic fact that if every subsequence of a sequence (x_n) has a subsequence with limit x then also $x_n \rightarrow x$) holds.

Lemma 6.29 *Let $A \in BDO(E)$ be a rich operator, $\Omega \subset \mathbb{R}^N$ be a valid starlike set, and take $\mathbb{I} \subseteq \mathbb{N}$. If every $\mathbb{J} \subseteq \mathbb{I}$ has a subset $\mathbb{K} \subseteq \mathbb{J}$ with $\mathbb{K} \in \text{suff}(A, \Omega)$ then $\mathbb{I} \in \text{suff}(A, \Omega)$.*

Proof. Contrarily to what we claim suppose $\mathbb{I} \notin \text{suff}(A, \Omega)$. Then $\sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ is elementwise but not uniformly invertible. By Theorem 6.21, $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{I}}$ is not stable. So there is a subset $\mathbb{J} = \{m_1, m_2, \dots\} \subseteq \mathbb{I}$ with

$$\|(\Pi_{m_j} A \Pi_{m_j} + \Theta_{m_j})^{-1}\| \geq j, \quad j = 1, 2, \dots$$

with the convention that $\|B^{-1}\| = \infty$ if B is not invertible. Hence, $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{J}}$ has no stable subsequence.

By our assumption, the index set $\mathbb{J} \subseteq \mathbb{I}$ has a subset $\mathbb{K} \in \text{suff}(A, \Omega)$. Since $\sigma_{\Omega, \mathbb{K}}^{\text{stab}}(A) \subseteq \sigma_{\Omega, \mathbb{I}}^{\text{stab}}(A)$ and all elements of the latter are invertible, we have that $\sigma_{\Omega, \mathbb{K}}^{\text{stab}}(A)$ is elementwise and hence uniformly invertible. By Theorem 6.21 again, $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{K}}$ is stable. But this contradicts the fact that $\mathbb{K} \subseteq \mathbb{J}$ and $(\Pi_n A \Pi_n + \Theta_n)_{n \in \mathbb{J}}$ has no stable subsequence. ■

Lemma 6.29 reduces the problem of showing that \mathbb{I} is sufficient to showing that every subset of \mathbb{I} has a sufficient subset \mathbb{K} . The new problem is about how to choose \mathbb{K} ; that is, one has to single out a subset $\mathbb{K} \subseteq \mathbb{J} \subseteq \mathbb{I}$ such that $\sigma_{\Omega, \mathbb{K}}^{\text{stab}}(A)$ is as small as possible (ideally finite or compact in some sense) in order to be uniformly invertible if elementwise invertible. This is exactly what one does in case $N = 1$ (see [145, Theorem 6]):

Proof of Proposition 6.26. Let $\mathbb{I} \subseteq \mathbb{N}$ and $\mathbb{J} \subseteq \mathbb{I}$ be arbitrary and let Ω be the interval (open, closed or semi-open) from $a < 0$ to $b > 0$. Since A is rich there is a subset $\mathbb{K} = \{k_1, k_2, \dots\} \subseteq \mathbb{J}$ such that both limit operators $C = A_h$ and $B = A_g$ exist, where $h = ([k_1 a], [k_2 a], \dots)$ tends to $-\infty$ and $g = ([k_1 b], [k_2 b], \dots)$ to $+\infty$, and hence

$$\sigma_{\Omega, \mathbb{K}}^{\text{stab}}(A) = \{ A, P_{\mathbb{N}_0} C P_{\mathbb{N}_0} + Q_{\mathbb{N}_0}, P_{-\mathbb{N}_0} B P_{-\mathbb{N}_0} + Q_{-\mathbb{N}_0} \} \quad (6.24)$$

is finite. So \mathbb{K} is sufficient w.r.t. A and Ω and thus, by Lemma 6.29, \mathbb{I} is sufficient. ■

For $N \geq 2$, a strategy might be to look at the partially ordered set of all $\sigma_{\Omega, \mathbb{K}}^{\text{stab}}(A)$ with $\mathbb{K} \subseteq \mathbb{J}$, ordered by inclusion, and to look for minimal elements. In case $N = 1$, these minimal elements consist of only three operators: A itself and one operator associated with each “direction” leading to infinity, as in (6.24). How can we capture this notion of “direction” in dimensions $N \geq 2$? In Example 6.23 we have seen that, for our purposes, it is not enough to associate a “direction” with each straight line from the origin to infinity; instead it seems one has to look at what are called admissible domains in the first symbol calculus of Rabinovich, Roch and Silbermann [140, 143] or, alternatively, at the Stone-Čech boundary of \mathbb{Z}^N as in the second symbol calculus by the same authors (also see [70, 155]).

6.4 Strategy 2: Rectangular Finite Sections

We try to think of a new idea for a stable truncation method. For simplicity, think of dimension $N = 1$ and $\Omega = [-1, 1]$ so that $\Omega_n = \{-n, \dots, n\}$ and hence $\Pi_n = P_n$ and $\Theta_n = Q_n$ for all $n \in \mathbb{N}$. The problem with the FSM becomes clear when we look at examples as simple as the shift operator $A = V_k$ with $k = 1$, say.

The FSM for the solution of $Au = b$, that is

$$P_n A P_n u_n = P_n b, \quad n = 1, 2, \dots, \quad (6.25)$$

thinks of an approximate solution u_n with support in $\{-n, \dots, n\}$, then applies the operator – in our case the forward shift by 1 component – and afterwards cuts off at $\{-n, \dots, n\}$ again, hereby trying to match the restriction of the right-hand side b to $\{-n, \dots, n\}$. It is clear that this truncated equation (6.25) is in general not solvable since the left-hand side of (6.25) always has a 0 at component $-n$ whereas the right-hand side has the same component $-n$ as b has. Even if $b(-n) = 0$ and (6.25) is solvable then the solution u_n is not unique² since its n th component got shifted and then cut off whence it is irrelevant for (6.25).

The observation generalises to band and band-dominated operators of course. If we truncate u_n at $\{-n, \dots, n\}$ and apply $A \in BO_w(E)$ then $AP_n u_n$ is supported in $\{-n-w, \dots, n+w\}$ whence, for the same reasons as illustrated for the shift $A = V_1$, it is better to cut off at $\{-m, \dots, m\}$ with $m = n + w$ and not at $\{-n, \dots, n\}$. The resulting system

$$P_m A P_n u_n = P_m b, \quad n = 1, 2, \dots, \quad (6.26)$$

with $m = n + w$ is over-determined – it has rectangular matrices that have $2w$ more rows than they have columns. But one can still try to solve it approximately (by least squares, say).

From the matrix point of view, $[AP_n]$ is the same as $[A]$, only with all but columns number $-n, \dots, n$ put to zero. If the horizontal cut-off $[P_m A P_n]$ (that one also has to do to get a finite system for the computer) is done at $m = n$, like in (6.25), then some ‘large’ entries of $[AP_n]$ will get cut off (recall $A = V_1$) which might cause problems as mentioned earlier; so it could be good to choose m a bit larger. In fact, if A has the property

$$P_m A P_n \rightrightarrows AP_n, \quad \text{i.e.} \quad Q_m A P_n \rightrightarrows 0 \quad \text{as} \quad m \rightarrow \infty \quad (6.27)$$

for all $n \in \mathbb{N}$ then it seems possible to work with this rectangular cut-off idea, where m in (6.26), depending on n , is chosen large enough to make $\|AP_n - P_m A P_n\| = \|Q_m A P_n\|$ small enough. The class of operators with property (6.27) clearly contains all of $L(E, \mathcal{P}) \supset BDO(E)$.

The above idea is so natural that it can hardly be new. Indeed, it is already used by much of the numerical community and it goes back at least to the 1960’s

²Of course, for a finite quadratic system, solvability for all right-hand sides (i.e. surjectivity of the finite matrix operator) is equivalent to uniqueness of the solution (i.e. injectivity). The approach here is to say that lack of surjectivity can be overcome by looking for approximate rather than exact solutions, whereas lack of injectivity is a more serious problem that will be dealt with by adding more equations (i.e. more matrix rows) to the finite system.

when Cleve Moler suggested, roughly speaking: If square submatrices give you problems, make them higher and use least squares.

In [82] (see Section 7.2.4 below) we have not only reinvented this method, we have (and that seems to be new) given a proof that, in the setting of a rather general Banach space E , the method is applicable as soon as A is invertible and subject to (6.27). There are no further conditions on the limit operators of A , etc. Of course, on the down side, for general operators A we do not really know yet how to choose m in dependence on n . However, the choice $m = n + w$ is clear for operators with band width w , and something similar is possible for a band-dominated operator A (where w must be fitted to the function f_A from (6.8), also see [106, p. 32ff]).

6.5 Comments and References

Polski's theorem (Theorem 6.3) for \mathcal{P} -convergent operator sequences with strictly convergent right-hand sides and solutions goes back to Roch and Silbermann [154] (also see [133] and Theorem 6.1.3 in [143]). Lemma 6.4 is from [106].

As mentioned before, in [103, 104, 106] there is a study of stability for approximation methods $(A_n)_{n \in \mathbb{R}_+}$, which creates technical subtleties (that can be dealt with but the treatment of which somehow distracts from our line of exposition here): Firstly, one loses Lemma 6.4. Secondly, $\oplus A_n$ now has layers enumerated by real numbers n and, as an operator on $E' = L^p(\mathbb{R}^{N+1})$ (assuming that $E = L^p(\mathbb{R}^N)$), $\oplus A_n$ is independent of every set of operators $\{A_n : n \in \mathbb{I}\}$ with index set $\mathbb{I} \subset \mathbb{R}_+$ of measure zero. The latter makes the topic of Section 6.1.3 technically more involved; in particular it is no longer possible (without further assumptions on the mapping $n \mapsto A_n$) to conclude stability of $(A_n)_{n \in \mathbb{R}_+}$ from invertibility at infinity of $\oplus A_n$.

The idea to identify the stability of a sequence of operators (A_n) with Fredholm properties of an associated block-diagonal (or 'stacked') operator $\oplus A_n$ was probably first made for the finite section method of Toeplitz operators with homogenous symbol. For $N = 1$ this goes back to Douglas and Howe [59], and for arbitrary N it was proven by Gorodetski [76] (see [101, 102] for extensions of their results).

The idea of the finite section method (FSM) is so natural that it is difficult to give a historical starting point. First rigorous treatments are from Baxter [16] and Gohberg & Feldman [71] on Wiener-Hopf and Toeplitz operators in the early 1960's. For the state of the art in the case of general band-dominated operators on $E = E^2(\mathbb{C})$, see [153].

The quest for stable subsequences if the FSM itself is instable is getting more attention recently [144, 145, 162, 163, 110]. Also the consideration of rectangular finite sections, although not new in the numerical community, is now gaining more focus in the numerical functional analysis literature (see [83, 164] for Toeplitz operators, [162, 163] for band-dominated operators and [82] for operators that are merely subject to the second condition in (3.6)).

Chapter 7

Applications

In this final chapter we will present applications of the theory derived above to concrete problems from mathematical physics. Applications largely divide into two classes: ‘Fredholm and Spectral Studies’ and ‘Approximation Methods’. So this is how we will subdivide this chapter. Instead of a separate section ‘Comments and References’ at the end, we here try to give some more references and background information as we go.

7.1 Fredholm and Spectral Studies

In this section we are looking at Fredholmness and invertibility, or equivalently: at essential spectrum and spectrum, of discrete Schrödinger operators [39, §7], a bi- [109] and a tridiagonal [32] random operator, and a class of integral operators on \mathbb{R}^N [106, §4.2], [39, §8], all of which are in general non-selfadjoint.

7.1.1 Discrete Schrödinger Operators

In this section we illustrate the results of Chapter 5, in particular the results of Section 5.3, in the relatively simple but practically relevant setting of $E = E^p(X)$ with $p \in \{0\} \cup [1, \infty]$ and a finite-dimensional space X . For applications to a class of operators on $E^p(X)$ with X infinite-dimensional, see e.g. Section 7.1.4.

We suppose that our operator A is a *discrete Schrödinger operator* on E in the sense e.g. of [51]. By this we mean that A is of the form

$$A = L + M_b$$

with a translation invariant operator L , i.e. $V_{-\alpha}LV_{\alpha} = L$ for all $\alpha \in \mathbb{Z}^N$, and

with a generalised multiplication operator M_b with $b \in E^\infty(L(X))$. A translation invariant operator L on E is a Laurent operator (see Example 3.62), and the sequence b is typically called the *potential* of A . The matrix representation of L is a *Laurent matrix* $[L] = (\lambda_{i-j})_{i,j \in \mathbb{Z}^N}$ with $\lambda_k \in L(X)$ for all $k \in \mathbb{Z}^N$. To be able to apply the results of Section 5.3 we will suppose that $A = L + M_b \in L(E^p)$, for $1 \leq p \leq \infty$, which is the case if $L \in \mathcal{W}$, i.e. if

$$\|L\|_{\mathcal{W}} = \sum_{k \in \mathbb{Z}^N} \|\lambda_k\| < \infty.$$

Discrete (or lattice) Schrödinger operators are widely studied in mathematical physics (see e.g. [148, §XI.14], [165], [26], [92], [172], [11]). Here are some examples of literature on spectral properties of Schrödinger and more general Jacobi operators where limit-operator-type arguments play an essential role: [128, 51, 3, 114, 67, 68, 69, 70, 137, 100, 99, 149, 150]. Much of this work is along the lines of formula (1.2) (often with the closure taken on the right-hand side); the three last papers also shed some light on the role of limit operators in the study of absolutely continuous spectrum.

Particularly common and classical is the case where the Laurent operator L takes the form

$$L = \sum_{k=1}^N (V_{e^{(k)}} + V_{-e^{(k)}}), \quad (7.1)$$

where $e^{(1)}, \dots, e^{(N)}$ are the unit coordinate vectors in \mathbb{Z}^N . The operator $A = L + M_b$ is then a discrete analogue of the second order differential operator $-\Delta + M$ where Δ is the Laplacian and M is the operator of multiplication by a bounded potential, both on \mathbb{R}^N .

Let L' be the Laurent operator with matrix representation $[L'] = (\lambda_{j-i}^*)_{i,j \in \mathbb{Z}^N}$. Then, identifying X with X^* (so that $X = X^* = X^\natural$), $L' \in \mathcal{W}$ and $A' := L' + M_b \in \mathcal{W}$. Further, in the case $p = \infty$, when we consider A as an operator on $E = E^\infty$, A' , considered as an operator on E^1 , is the unique transpose of A with respect to the dual system (E^∞, E^1) of Section 3.3.2 and so the unique preadjoint of A , for it holds for $u \in E$, $v \in E^1$, using equations (3.15) and (3.14), that

$$(Au)(v) = (Au, v) = (u, A'v) = u(A'v).$$

We will say that A is *symmetric* if $A = A'$. For example, this is the case when L is the classical operator (7.1).

For $b \in E^\infty(L(X))$ let $\text{Lim}(b)$ denote the set of *limit functions* of b , by which we mean the set of all functions $b_h \in E^\infty(L(X))$ for which there exists a sequence $h = (h(n))_{n \in \mathbb{N}} \subset \mathbb{Z}^N$ tending to infinity such that

$$b_h(m) = \lim_{n \rightarrow \infty} b(m + h(n)), \quad m \in \mathbb{Z}^N. \quad (7.2)$$

It follows from (3.25) that

$$\sigma^{\text{op}}(A) = \{L + M_c : c \in \text{Lim}(b)\}.$$

Noting that Corollary 5.24 applies to $A = L + M_b$, we have the following result. In this result, for an operator $B \in \mathcal{W}$, we denote by $\text{spec}^p(B)$, $\text{spec}_{\text{ess}}^p(B)$, and $\text{spec}_{\text{point}}^p(B)$, respectively, the spectrum, essential spectrum, and point spectrum (set of eigenvalues) of B considered as an operator on E^p .

Theorem 7.1 *The following statements are equivalent:*

- (a) $L + M_c$ is injective on E^∞ for all $c \in \text{Lim}(b)$ and, for some s -dense subset $\varsigma \subset \text{Lim}(b)$, $L' + M_c$ is injective on E^1 for all $c \in \varsigma$.
- (b) $L + M_c$ is invertible, for every $c \in \text{Lim}(b)$, on one of the spaces E^p ;
- (c) $L + M_c$ is invertible on E^p for every p and every $c \in \text{Lim}(b)$ and the inverses are uniformly bounded (in p and c);
- (d) A is Fredholm on one of the spaces E^p ;
- (e) A is Fredholm on all of the spaces E^p and the index is the same on each space.

Thus, for every p it holds that

$$\text{spec}_{\text{ess}}^p(A) = \bigcup_{c \in \text{Lim}(b)} \text{spec}^p(L + M_c) \quad (7.3)$$

$$= \bigcup_{c \in \text{Lim}(b)} [\text{spec}_{\text{point}}^\infty(L + M_c) \cup \text{spec}_{\text{point}}^1(L' + M_c)] \quad (7.4)$$

and

$$\text{spec}^p(A) = \text{spec}_{\text{point}}^\infty(A) \cup \text{spec}_{\text{point}}^1(A') \cup \text{spec}_{\text{ess}}^p(A) \quad (7.5)$$

$$= \text{spec}_{\text{point}}^1(A) \cup \text{spec}_{\text{point}}^1(A') \cup \text{spec}_{\text{ess}}^p(A). \quad (7.6)$$

Proof. From the equivalence of (a) and (b) in Corollary 5.11 we have that (a) is equivalent to the statement that $\alpha(L + M_c) = 0$ for all $c \in \text{Lim}(b)$ and $\beta(L + M_c) = 0$ for all $c \in \varsigma$. By Corollary 5.24 this is equivalent to (b)–(e). So it remains to prove (7.5)+(7.6). Equality (7.5) follows since, as noted after Lemma 5.18, the spectrum of A does not depend on p , so that $\text{spec}^p(A) = \text{spec}^\infty(A)$. Further, if $\lambda \in \text{spec}^\infty(A)$ and $\lambda I - A$ is Fredholm, then either $\alpha(\lambda I - A) \neq 0$ or $\beta(\lambda I - A) \neq 0$, so that either $\lambda \in \text{spec}_{\text{point}}^\infty(A)$ or $\lambda \in \text{spec}_{\text{point}}^1(A')$. To see equality (7.6), note

that $\text{spec}_{\text{point}}^1(A) \subset \text{spec}_{\text{point}}^\infty(A)$ since injectivity of $\lambda I - A$ on E^∞ implies its injectivity on $E^1 \subset E^\infty$. Moreover, if $\lambda I - A$ is Fredholm then, by Proposition 3.37, the kernel of $\lambda I - A$ is a subset of E^0 . Since $(\lambda I - A)|_{E^0}$ is Fredholm with the same index on E^0 and $E^1 \subset E^0$ and since E^1 is dense in E^0 , it follows from a standard result on Fredholm operators (e.g. [133]) that the kernel of $\lambda I - A$ is a subset of E^1 . Thus $\text{spec}_{\text{point}}^\infty(A) \subset \text{spec}_{\text{ess}}^\infty(A) \cup \text{spec}_{\text{point}}^1(A)$. ■

Remark 7.2 We note that main parts of the above result, namely equality (7.3) and that the spectrum and essential spectrum do not depend on $p \in [1, \infty]$, are well known (see e.g. [143, Theorem 5.8.1]). The characterisation of the essential spectrum by (7.4) appears to be new. □

Clearly, equations (7.3) – (7.6) simplify when L is symmetric, for example if L is given by (7.1), since we then have that $\text{spec}_{\text{point}}^1(A') = \text{spec}_{\text{point}}^1(A)$ and $\text{spec}_{\text{point}}^1(L' + M_c) \subset \text{spec}_{\text{point}}^\infty(L + M_c)$ for all $c \in \text{Lim}(b)$. Simplifications also occur when the potential b is almost periodic, $b \in E_{\text{AP}}^\infty(L(X))$, in which case $\text{Lim}(b)$ is precisely what is often called the *hull* of b , the set $\text{clos}_{E^\infty(L(X))}\{V_k b : k \in \mathbb{Z}^N\}$, the closure of the set of translates of b .

Theorem 7.3 *If b is almost periodic then, for all p and all $\tilde{b} \in \text{Lim}(b)$,*

$$\text{spec}_{\text{ess}}^p(A) = \text{spec}^p(A) = \text{spec}^p(L + M_{\tilde{b}}) = \bigcup_{c \in \text{Lim}(b)} \text{spec}_{\text{point}}^\infty(L + M_c). \quad (7.7)$$

Proof. L is translation invariant and hence almost periodic. Since b is almost periodic, M_b is almost periodic by Lemma 5.43. Thus A is almost periodic. Further, $\sigma^{\text{op}}(A - I)$ is uniformly bounded by Theorem 4.26 (i) and so uniformly Montel on E^∞ by Corollary 4.17, since $\dim X < \infty$. The result thus follows from Theorem 5.44, Theorem 5.40 (iv), and the equivalence of statements (b) and (d) in Theorem 7.1. ■

Remark 7.4 That $\text{spec}_{\text{ess}}^p(A) = \text{spec}^p(A) = \text{spec}^p(L + M_{\tilde{b}})$ for all $\tilde{b} \in \text{Lim}(b)$, the hull of b , is a classical result, see e.g. [165, 172, 143]. The result that

$$\text{spec}^p(A) = \bigcup_{c \in \text{Lim}(b)} \text{spec}_{\text{point}}^\infty(L + M_c)$$

appears to be new in this generality. However, Avila and Jitomirskaya have a comparable result for the case when A is self-adjoint (i.e. $L = L'$ and the potential b is real-valued) on $E = \ell^2(\mathbb{Z}, \mathbb{C})$ in their very recent paper [14] (see Theorem 3.3 there). Moreover, analogous results for uniformly elliptic differential operators on \mathbb{R}^N with almost periodic coefficients date back to Shubin [170].

Moreover, note that this result is well-known, as a part of Floquet-Bloch theory [92, 93, 53], in the case when b is periodic; in fact one has the stronger result in

that case, at least when L is given by (7.1), that λ is in the spectrum of A iff there exists a solution $u \in E^\infty$ of $\lambda u = Au$ which is quasi-periodic in the sense of [92]. The latter means that $u(m) = \exp(ik \cdot m)y(m)$ for all $m \in \mathbb{Z}^N$, where $y \in E^\infty$ is periodic and $k \in \mathbb{R}^N$ is fixed, so that if u is quasi-periodic then it is certainly almost periodic. Thus, if b is periodic then λ is in the spectrum of A iff there exists a solution $u \in E^\infty$ of $\lambda u = Au$ which is almost periodic.

Natural questions are whether this statement still holds for the case when b is almost periodic, at least for L given by (7.1), or whether the weaker statement holds that λ is in the spectrum of A iff, for some $c \in \text{Lim}(b)$, there exists an almost periodic solution $u \in E^\infty$ of $\lambda u = (L + M_c)u$. The answer is, to our knowledge, unknown. We would however like to mention that there are examples (see [49, 131] and [128, p. 454]) of almost periodic potentials b in the case $N = 1$ and with L given by (7.1) such that, for almost¹ all $c \in \text{Lim}(b)$, it holds that all bounded eigenfunctions of $L + M_c$ decay exponentially (and therefore are not almost periodic, of course). \square

To illustrate the application of the above theorem in the 1D case ($N = 1$) we consider a widely studied class of almost periodic operators obtained by the following construction. For some $d \in \mathbb{N}$ let $B : \mathbb{R}^d \rightarrow L(X)$ be a continuous function satisfying

$$B(s + m) = B(s), \quad s \in \mathbb{R}^d, \quad m \in \mathbb{Z}^d.$$

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ and, for $s \in \mathbb{R}^d$ let $b_s : \mathbb{Z} \rightarrow L(X)$ be given by

$$b_s(n) = B(\alpha n + s), \quad n \in \mathbb{Z}. \quad (7.8)$$

If $\alpha_1, \dots, \alpha_d$ are all rational, then b_s is periodic. Whatever the choice of $\alpha_1, \dots, \alpha_d$, b_s is almost periodic ($b_s \in E_{\text{AP}}^\infty(L(X))$).

For $s \in \mathbb{R}^d$ let $[s]$ denote the coset $[s] = s + \mathbb{Z}^d$ in $\mathbb{R}^d/\mathbb{Z}^d$. An interesting case is that in which $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are rationally independent, in which case $\{[\alpha m] : m \in \mathbb{Z}\}$ is dense in $\mathbb{R}^d/\mathbb{Z}^d$. Then it is a straightforward calculation to see that

$$\text{Lim}(b_s) = \{b_t : t \in \mathbb{R}^d\}. \quad (7.9)$$

Thus, for this case, (7.7) reads as

$$\text{spec}_{\text{ess}}^p(L + M_{b_s}) = \text{spec}^p(L + M_{b_s}) = \bigcup_{t \in \mathbb{R}^d} \text{spec}_{\text{point}}^\infty(L + M_{b_t}). \quad (7.10)$$

As a particular instance, this formula holds in the case when $X = \mathbb{C}$, $d = 1$, and $B(s) = \lambda \cos(2\pi s)$, $s \in \mathbb{R}$, for some $\lambda \in \mathbb{C}$. Then

$$b_s(n) = \lambda \cos(2\pi(\alpha n + s)), \quad n \in \mathbb{Z}, \quad (7.11)$$

¹namely those $c \in \text{Lim}(b)$ for which a certain non-resonance condition holds for $L + M_c$

and (7.10) holds if α is irrational, in which case b_s is the so-called *almost Mathieu* potential.

We next modify the above example to illustrate the application of Theorem 7.1 in a particular 1D ($N = 1$) case.

Example 7.5 Define $b_s \in E_{\text{AP}}^\infty(L(X))$ by (7.8) and suppose that $1, \alpha_1, \alpha_2, \dots, \alpha_d$ are rationally independent. Suppose that $f : \mathbb{Z} \rightarrow \mathbb{R}^d$ is slowly oscillating, i.e. f satisfies

$$\lim_{|n| \rightarrow \infty} |f(n+1) - f(n)| \rightarrow 0.$$

Define $b \in E^\infty(L(X))$ by

$$b(n) = B(\alpha n + f(n)), \quad n \in \mathbb{Z}.$$

Then it is straightforward to see that $\text{Lim}(b) \subset \{b_s : s \in \mathbb{R}^d\}$. Since, by Theorem 4.26 (iv), $b_s \in \text{Lim}(b)$ implies that $\text{Lim}(b_s) \subset \text{Lim}(b)$, we have, by (7.9), that

$$\text{Lim}(b) = \{b_s : s \in \mathbb{R}^d\}.$$

Thus, applying Theorem 7.1 and (7.10), we see that, for every $s \in \mathbb{R}^d$ and every $p \in \{0\} \cup [1, \infty]$,

$$\text{spec}_{\text{ess}}^p(L + M_b) = \text{spec}_{\text{ess}}^p(L + M_{b_s}) = \text{spec}^p(L + M_{b_s}) = \bigcup_{t \in \mathbb{R}^d} \text{spec}_{\text{point}}^\infty(L + M_{b_t}). \quad (7.12)$$

We note that, in the special case that L is given by (7.1) (with $N = 1$), $X = \mathbb{C}$, and B is real-valued, the statement that

$$\text{spec}_{\text{ess}}^2(L + M_b) = \text{spec}^2(L + M_{b_s})$$

for all $s \in \mathbb{R}^d$ is Theorem 5.2 of Last and Simon [100] (established by limit operator type arguments). As a specific instance where (7.12) holds, let us take $X = \mathbb{C}$, $d = 1$, and $B(s) = \lambda \cos(2\pi s)$, $s \in \mathbb{R}$, for some $\lambda \in \mathbb{C}$. Then b_s is given by (7.11) and, taking (as one possible choice), $f(n) = |n|^{1/2}$, one has

$$b(n) = \lambda \cos(2\pi(\alpha n + |n|^{1/2}))$$

(cf. [100, Theorem 1.3]). \square

As a further example we consider the case when b is pseudoergodic as introduced in Section 5.5.3. From Corollary 5.52 we get that b is pseudoergodic iff $\text{Lim}(b)$ is the set $\Sigma^{\mathbb{Z}^N}$ of all functions $c : \mathbb{Z}^N \rightarrow \Sigma$. In particular, $b \in \text{Lim}(b)$ if b is pseudoergodic.

Theorem 7.6 *If b is pseudoergodic then, for all p ,*

$$\operatorname{spec}_{\text{ess}}^p(A) = \operatorname{spec}^p(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \operatorname{spec}^p(L + M_c) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \operatorname{spec}_{\text{point}}^\infty(L + M_c).$$

Proof. The first two ‘=’ signs follow from (7.3) and the fact that $b \in \operatorname{Lim}(b) = \Sigma^{\mathbb{Z}^N}$. For the proof of the remaining equality, we refer to the following s -dense subset of $\operatorname{Lim}(b) = \Sigma^{\mathbb{Z}^N}$: Let $m_1 = m_2 = \dots = m_N = 1$, and let ς stand for the set of all periodic functions $u : \mathbb{Z}^N \rightarrow \Sigma$, that is

$$\varsigma := \bigcup_{n \in \mathbb{N}} E_n^\infty(\Sigma)$$

with $E_n^\infty(\Sigma)$ defined as in (5.20) (with the slight abuse of notation by writing $E^\infty(\Sigma)$ for $\Sigma^{\mathbb{Z}^N}$, i.e. the set of all functions $u : \mathbb{Z}^N \rightarrow \Sigma$). Then ς is s -dense in $\Sigma^{\mathbb{Z}^N}$ as every $u \in \Sigma^{\mathbb{Z}^N}$ can be strictly approximated by the sequence $(\tilde{P}_n u) \subset \varsigma$ with \tilde{P}_n as defined in (5.21). If $\lambda \in \mathbb{C}$ and all limit operators $\lambda I - (L + M_c)$ of $\lambda I - A = \lambda I - (L + M_b)$, including those with $c \in \varsigma$, are injective, then, by Theorem 5.37, we have that $\lambda I - (L + M_c)$ is surjective for every $c \in \varsigma$. By the equivalence between (a) and (d) in Theorem 7.1, this shows that $\lambda I - A = \lambda I - (L + M_b)$ is Fredholm. ■

Remark 7.7 It is shown that

$$\operatorname{spec}_{\text{ess}}^2(A) = \operatorname{spec}^2(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \operatorname{spec}^2(L + M_c)$$

in [51]. The result that $\operatorname{spec}^p(A) = \bigcup_{c \in \Sigma^{\mathbb{Z}^N}} \operatorname{spec}_{\text{point}}^\infty(L + M_c)$ appears to be new. □

The above theorems show that, in each of the cases L symmetric, b almost periodic, and b pseudoergodic, it holds that

$$\operatorname{spec}_{\text{ess}}^p(A) = \bigcup_{c \in \operatorname{Lim}(c)} \operatorname{spec}_{\text{point}}^\infty(L + M_c). \quad (7.13)$$

We conjecture that, in fact, this equation holds for all $c \in E^\infty(L(X))$. For $N = 1$ this is no longer a conjecture, as we showed in Corollary 5.26 (which follows from our more general results in [38], also see Theorem 5.12 above). For $N \geq 2$ however, this is an open problem.

We finish this section with an example of how Theorem 7.6 can be used to compute spectra of Schrödinger operators with random potential b .

Example 7.8 Let $N = 1$, $p \in [1, \infty]$, $X = \mathbb{C}$ and take a compact set Σ in the complex plane. We compute the spectrum of $A = L + M_b$ as an operator on

$E = E^p(X)$ where $L = V_{-1}$ is the backward shift and the function values $b(k)$, $k \in \mathbb{Z}$, of the random potential b are chosen independently of each other from the set Σ . Under the same conditions as in Example 5.49, one gets that, with probability 1, b is pseudoergodic.

The calculation of the point spectra in Theorem 7.6 is a special case of the calculations demonstrated below in Section 7.1.2. Applying the theorem, we get that, with probability 1,

$$\text{spec}^p A = \text{spec}_{\text{ess}}^p A = \bigcup_{c \in \Sigma^{\mathbb{Z}}} \text{spec}_{\text{point}}^{\infty}(L + M_c) = \Sigma_{\cup}^1 \setminus \Sigma_{\cap}^1 \quad (7.14)$$

where

$$\Sigma_{\cup}^1 := \bigcup_{\sigma \in \Sigma} (\sigma + \overline{\mathbb{D}}) \quad \text{and} \quad \Sigma_{\cap}^1 := \bigcap_{\sigma \in \Sigma} (\sigma + \mathbb{D})$$

with \mathbb{D} denoting the open unit disk in \mathbb{C} and $\overline{\mathbb{D}}$ its closure.

Formula (7.14) confirms, in a simpler and more straightforward way, a result of Trefethen, Contedini and Embree [175, Theorem 8.1] (and see [176, Section VIII]). Equation (7.14) is illustrated, for two particular cases, in Figure 7.1. \square

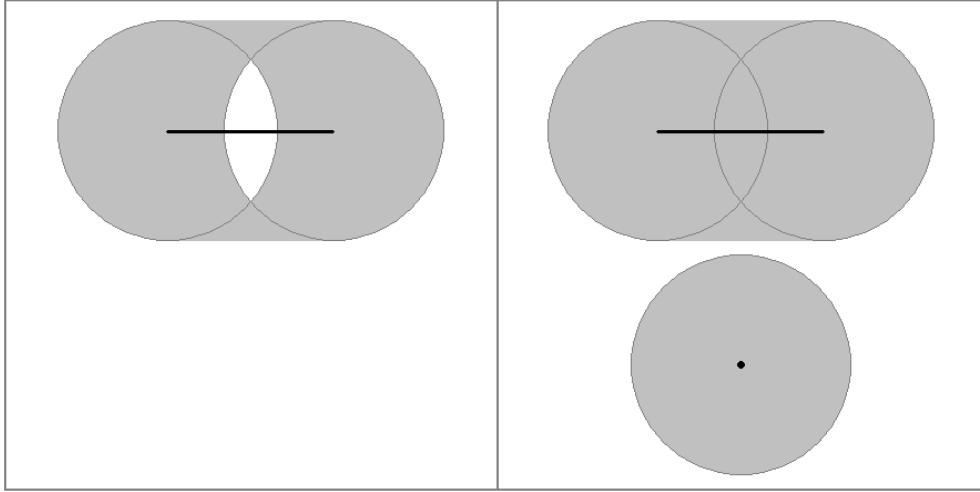


Figure 7.1: The left image shows, as a gray shaded area, $\text{spec}^p A$ when Σ is the black straight line of length 1.5. In the right image, one more point (the centre of the lower circle) has been added to Σ which results in $\Sigma_{\cap}^1 = \emptyset$.

7.1.2 A Bidigonal Random Matrix

Given two compact sets Σ and \mathcal{T} in the complex plane, we study the spectrum of the bidiagonal infinite matrix

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & \sigma_{-2} & \tau_{-2} & & & \\ & & \sigma_{-1} & \tau_{-1} & & \\ & & & \sigma_0 & \tau_0 & \\ & & & & \sigma_1 & \tau_1 \\ & & & & & \sigma_2 & \ddots \\ & & & & & & \ddots \end{pmatrix}, \quad (7.15)$$

considered as an operator A on $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$, where $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ are independent samples from two random variables, X and Y , with values in Σ and \mathcal{T} , respectively. Again we will suppose that, for all $\varepsilon > 0$, $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$, the probabilities of $|X - \sigma| < \varepsilon$ and $|Y - \tau| < \varepsilon$ are both nonzero.

We know from Corollary 3.66 and Theorem 3.68 that both $\text{spec}^p A$ and $\text{spec}_{\text{ess}}^p A$ are independent of p since $A \in BO_1(E) \subset \mathcal{W}$.

Matrices like (7.15) and the question about their spectra originate from problems in so-called non-selfadjoint quantum mechanics. For example, they appear as Hamiltonians of asymmetric randomly hopping quantum particles, where, in the case when $\Sigma = \{-1, 1\}$ and $\mathcal{T} = \{1\}$, (7.15) is called the “one-way model” by Brézin, Feinberg and Zee [24, 64, 65].

Before we can state the result, let us put, for $\varepsilon \geq 0$,

$$\Sigma_{\cup}^{\varepsilon} := \bigcup_{\sigma \in \Sigma} \overline{U}_{\varepsilon}(\sigma), \quad \text{and} \quad \Sigma_{\cap}^{\varepsilon} := \bigcap_{\sigma \in \Sigma} U_{\varepsilon}(\sigma)$$

with $U_{\varepsilon}(\sigma) = \{\lambda \in \mathbb{C} : |\lambda - \sigma| < \varepsilon\}$ and $\overline{U}_{\varepsilon}(\sigma) = \{\lambda \in \mathbb{C} : |\lambda - \sigma| \leq \varepsilon\}$ denoting the open and the closed ε -neighbourhood of σ in \mathbb{C} , respectively. Then our result reads as follows:

Theorem 7.9 *If A is induced by the random matrix shown in (7.15) then, with probability 1,*

$$\text{spec } A = \text{spec}_{\text{ess}} A = \Sigma_{\cup}^T \setminus \Sigma_{\cap}^t,$$

where $T = \max\{|\tau| : \tau \in \mathcal{T}\}$ and $t = \min\{|\tau| : \tau \in \mathcal{T}\}$.

So in particular, the spectrum of A only depends on the supports Σ and \mathcal{T} of our probability distributions and not on the distributions themselves. With

Theorem 7.9 we generalize Theorem 8.1 of Trefethen, Contedini and Embree's paper [175] (also see [176, Section VIII]) where $\mathcal{T} = \{1\}$ and therefore $T = t = 1$. Also note that the case of a constant superdiagonal, i.e. when \mathcal{T} is a singleton, is a random discrete Schrödinger operator in the sense of the previous section and is briefly discussed in Example 7.8.

If we put $\Sigma = \{\sigma\}$ and $\mathcal{T} = \{\tau\}$ with $\sigma, \tau \in \mathbb{C}$ fixed then (7.15) is a Laurent matrix with two constant diagonals of value σ and τ , and Theorem 7.9 resembles the well-known fact (see e.g. [21]) that $\text{spec} A = \text{spec}_{\text{ess}} A$ is the circle of radius $T = t = |\tau|$ around σ .

If again, $\Sigma = \{\sigma\}$ is a singleton and \mathcal{T} consists of at least two points with different moduli $|\tau| \in [t, T]$ then letting $t = \min |\tau| \rightarrow 0$ in Theorem 7.9 demonstrates what is called the “disk-annulus transition” in e.g. [63, 66].

Another observation is that, if $\Sigma, \mathcal{T} \subset \mathbb{C}$ are compact sets and $t = \text{dist}(\mathcal{T}, 0)$ is small enough for $\Sigma_{\cap}^t = \emptyset$ (e.g. when $t \in [0, \text{diam } \Sigma/2]$) then we get that $\text{spec} A = \Sigma_{\cup}^T$ coincides with the ε -pseudospectrum, for $\varepsilon = T$, of the diagonal matrix that results from (7.15) by deleting the 1st superdiagonal.

We would also like to mention that, as expected for a non-symmetric matrix, the spectrum of A differs almost surely (unless $\mathcal{T} = \{0\}$, i.e. the symmetric case) from the limit as $n \rightarrow \infty$ of the spectra of its n -by- n finite sections, which obviously is Σ . This situation changes however if one more entry from \mathcal{T} is added to the n -by- n finite section of A at the corner position $(n, 1)$ to make the problem periodic.

If A is our random operator induced by (7.15) then, with probability 1, A is pseudoergodic in the sense of Section 5.5.3, i.e. $k \mapsto (\sigma_k, \tau_k)$ is a pseudoergodic mapping $\mathbb{Z} \rightarrow \Sigma \times \mathcal{T}$, so that

$$\begin{aligned} \sigma^{\text{op}}(A) \text{ is the set of } \mathbf{all} \text{ operators/matrices of the form (7.15)} \\ \text{with } \sigma_k \in \Sigma \text{ and } \tau_k \in \mathcal{T} \text{ for all } k \in \mathbb{Z}. \end{aligned} \quad (7.16)$$

So in particular, $A \in \sigma^{\text{op}}(A)$, which shows that, $\text{spec} A \subset \text{spec}_{\text{ess}} A$ and hence, by Corollary 5.26,

$$\text{spec } A = \text{spec}_{\text{ess}} A = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec } B = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}_{\text{point}}^{\infty} B. \quad (7.17)$$

The proof of Theorem 7.9 now rests on (7.17) and the above description (7.16) of the operator spectrum $\sigma^{\text{op}}(A)$.

Remark 7.10 Limit operator ideas and the validity of the first two “=” signs in (7.17) are not new in the spectral theory of random matrices (see e.g. [26, 52, 51, 75, 128]), but what seems to be new here is the third “=” sign in (7.17), due

to our results in Chapter 5 (and see [38] and [39]), and hence the possibility of the simple proof that is presented here. \square

Proof of Theorem 7.9. Let A be induced by the random matrix (7.15) with samples $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ from the probability distributions on the compact sets $\Sigma, \mathcal{T} \subset \mathbb{C}$ as described above, and put $T = \max\{|\tau| : \tau \in \mathcal{T}\}$ and $t = \min\{|\tau| : \tau \in \mathcal{T}\}$.

For the calculation of the point spectra in (7.17), take $\lambda \in \mathbb{C}$ and let $B \in \sigma^{\text{op}}(A)$, i.e. B is of the form (7.15) with $\sigma_k \in \Sigma$ and $\tau_k \in \mathcal{T}$ for all $k \in \mathbb{Z}$, by (7.16). If $u : \mathbb{Z} \rightarrow \mathbb{C}$ is a nontrivial solution of $Bu = \lambda u$ then $u(n_0) \neq 0$ for some $n_0 \in \mathbb{Z}$, w.l.o.g. let $u(n_0) = 1$, and

$$\tau_k u(k+1) = (\lambda - \sigma_k) u(k) \quad \text{for all } k \in \mathbb{Z}. \quad (7.18)$$

Case 1: $0 \notin \mathcal{T}$, i.e. $t > 0$.

Note that, by (7.18), $\lambda \neq \sigma_k$ for all $k < n_0$ since otherwise $u(n_0) = 0$ (recall that $\tau_k \neq 0$ for all k). As a consequence we get that

$$u(n) = \begin{cases} \prod_{k=n_0}^{n-1} \frac{\lambda - \sigma_k}{\tau_k}, & n \geq n_0, \\ \prod_{k=n}^{n_0-1} \frac{\tau_k}{\lambda - \sigma_k}, & n < n_0 \end{cases} \quad (7.19)$$

for every $n \in \mathbb{Z}$.

Clearly, if $\lambda \notin \Sigma_{\cup}^T$ then $|\lambda - \sigma| > T \geq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ and hence, for every nontrivial solution u of $Bu = \lambda u$, we have that $|u(n)| \rightarrow \infty$ in (7.19) as $n \rightarrow +\infty$ since $|\frac{\lambda - \sigma_k}{\tau_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B .

Similarly, if $\lambda \in \Sigma_{\cap}^t$ then $|\lambda - \sigma| < t \leq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ and hence, for every nontrivial solution u of $Bu = \lambda u$, $|u(n)| \rightarrow \infty$ in (7.19) as $n \rightarrow -\infty$ since $|\frac{\tau_k}{\lambda - \sigma_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B . (Note that n_0 in (7.19) depends on B and λ .)

So in both cases, $Bu = \lambda u$ has no nontrivial solution $u \in \ell^\infty(\mathbb{Z})$, so $\lambda \notin \text{spec}_{\text{point}}^\infty B$ for all $B \in \sigma^{\text{op}}(A)$ and hence, by (7.17), $\lambda \notin \text{spec} A$. Now it remains to look at $\lambda \in \Sigma_{\cup}^T \setminus \Sigma_{\cap}^t$. In this case, let $\sigma^*, \sigma_* \in \Sigma$ and $\tau^*, \tau_* \in \mathcal{T}$ be such that $|\lambda - \sigma_*| \leq |\tau^*|$ and $|\lambda - \sigma^*| \geq |\tau_*|$, which is possible by the choice of λ . Now

consider

$$B = \begin{pmatrix} \ddots & \ddots & & & & & \\ & \sigma^* & \tau_* & & & & \\ & & \sigma^* & \tau_* & & & \\ & & & \sigma_* & \tau^* & & \\ & & & & \sigma_* & \tau^* & \\ & & & & & \sigma_* & \tau^* \\ & & & & & & \ddots \\ & & & & & & & \ddots \end{pmatrix} \in \sigma^{\text{op}}(A)$$

with σ_* and τ^* in row $0, 1, 2, \dots$ and σ^* and τ_* in row $-1, -2, \dots$ to see that

$$u = \left(\dots, \left(\frac{\tau_*}{\lambda - \sigma^*} \right)^2, \left(\frac{\tau_*}{\lambda - \sigma^*} \right)^1, 1, \left(\frac{\lambda - \sigma_*}{\tau^*} \right)^1, \left(\frac{\lambda - \sigma_*}{\tau^*} \right)^2, \dots \right)^\top,$$

with the 1 at position $n_0 = 0$, is an eigenvector in $\ell^\infty(\mathbb{Z})$ of B w.r.t. λ . So we have that $\lambda \in \text{spec}_{\text{point}}^\infty B \subset \text{spec} A$, by (7.17).

Summarizing, we see that the formula in Theorem 7.9 holds in Case 1.

Case 2: $0 \in \mathcal{T}$ with $0 = t < T$, i.e. \mathcal{T} has points other than 0.

Suppose $\lambda \notin \Sigma_\cup^T$. Then $\lambda \neq \sigma_k$ for all $k \in \mathbb{Z}$ and, by (7.18), $\tau_k \neq 0$ for all $k \geq n_0$ since otherwise $u(n_0) = 0$. So again, (7.19) holds for all $n \in \mathbb{Z}$. But from $|\lambda - \sigma| > T \geq |\tau|$ for all $\sigma \in \Sigma$ and $\tau \in \mathcal{T}$ we again get that $|u(n)| \rightarrow \infty$ in (7.19) as $n \rightarrow +\infty$ since $|\frac{\lambda - \sigma_k}{\tau_k}| > 1$ for all $k \in \mathbb{Z}$, regardless of the particular entries σ_k and τ_k of B .

Now suppose $\lambda \in \Sigma_\cup^T$. Then fix $\tau \in \mathcal{T}$ with maximal modulus, i.e. $|\tau| = T > 0$ and take a $\sigma \in \Sigma$ with $|\lambda - \sigma| \leq T = |\tau|$. Now

$$B = \begin{pmatrix} \ddots & \ddots & & & & & \\ & \sigma & 0 & & & & \\ & & \sigma & 0 & & & \\ & & & \sigma & \tau & & \\ & & & & \sigma & \tau & \\ & & & & & \sigma & \tau \\ & & & & & & \ddots \\ & & & & & & & \ddots \end{pmatrix} \in \sigma^{\text{op}}(A)$$

with τ in row $0, 1, 2, \dots$ and 0 in row $-1, -2, \dots$ has

$$u = \left(\dots, 0, 0, 1, \left(\frac{\lambda - \sigma}{\tau} \right)^1, \left(\frac{\lambda - \sigma}{\tau} \right)^2, \dots \right)^\top \in \ell^\infty(\mathbb{Z}),$$

with the 1 at position $n_0 = 0$, as its eigenvector w.r.t. λ . So $\lambda \in \text{spec}_{\text{point}}^\infty B \subset \text{spec} A$, by (7.17).

So in Case 2 we get $\text{spec} A = \Sigma_\cup^T$. But the latter is equal to $\Sigma_\cup^T \setminus \Sigma_\cap^t$ since $t = 0$ and $\Sigma_\cap^0 = \emptyset$.

Case 3: $0 \in \mathcal{T}$ with $0 = t = T$, i.e. $\mathcal{T} = \{0\}$.

In this trivial case, A is a diagonal matrix, so that, with probability 1, $\text{spec} A = \Sigma$. But $\Sigma = \Sigma_\cup^T \setminus \Sigma_\cap^t$ if $T = t = 0$. ■

An a-posteriori Experiment:

Is it Enough to Look at Periodic Limit Operators?

Recall formula (7.17) for the spectrum of a bi-infinite, pseudoergodic and banded matrix operator A . Generally it is difficult to evaluate the rightmost term in (7.17) since the index set $\sigma^{\text{op}}(A)$ of this union is a very large set and the point spectrum of most operators $B \in \sigma^{\text{op}}(A)$ is difficult to determine. An approach which has been used by Davies and co-workers (see e.g. [51, 53, 115] and references therein) for studying the spectrum of such an operator A is to look at a large number of **periodic** limit operators B of A . More precisely, one looks at the subsets

$$\text{spec}_{\text{per}}^n A := \bigcup_{B \in \mathcal{P}_n(A)} \text{spec}_{\text{point}}^\infty B \quad \text{of} \quad \text{spec} A = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}_{\text{point}}^\infty B$$

for large values of $n \in \mathbb{N}$, where $\mathcal{P}_n(A) \subset \sigma^{\text{op}}(A)$ denotes the set of all limit operators of A with n -periodic diagonals. For $B \in \mathcal{P}_n(A)$, spectrum and ℓ^∞ point spectrum coincide (see our Theorem 5.37) and its computation reduces to the computation of the spectra of certain finite matrices by treating B as a block Laurent matrix with n -by- n block entries (see e.g. [21, 53, 115]).

An interesting question is under what circumstances the left-hand side of the inclusion

$$\text{spec}_{\text{per}} A := \bigcup_{n=1}^{\infty} \text{spec}_{\text{per}}^n A \subset \text{spec} A \quad (7.20)$$

is dense in the right-hand side. In this section we illustrate that, even when the pseudoergodic operator A is non-normal, it can happen that the closure of the left-hand side of (7.20) is equal to the spectrum of A .

To do this, we will look at Brézin, Feinberg and Zee's “one-way model” (7.15),

where $\Sigma = \{-1, 1\}$ and $\mathcal{T} = \{1\}$; that is when A is induced by

$$\begin{pmatrix} \ddots & \ddots & & & & \\ & \sigma_{-1} & 1 & & & \\ & & \sigma_0 & 1 & & \\ & & & \sigma_1 & 1 & \\ & & & & \sigma_2 & \ddots \\ & & & & & \ddots \end{pmatrix} \quad (7.21)$$

with σ_k randomly chosen from $\Sigma = \{-1, 1\}$. The spectrum of A is explicitly known due to [175], from our Example 7.8 or from Theorem 7.9: It is the union of the two disks of radius 1 centered at 1 and -1 (see Figure 7.2).

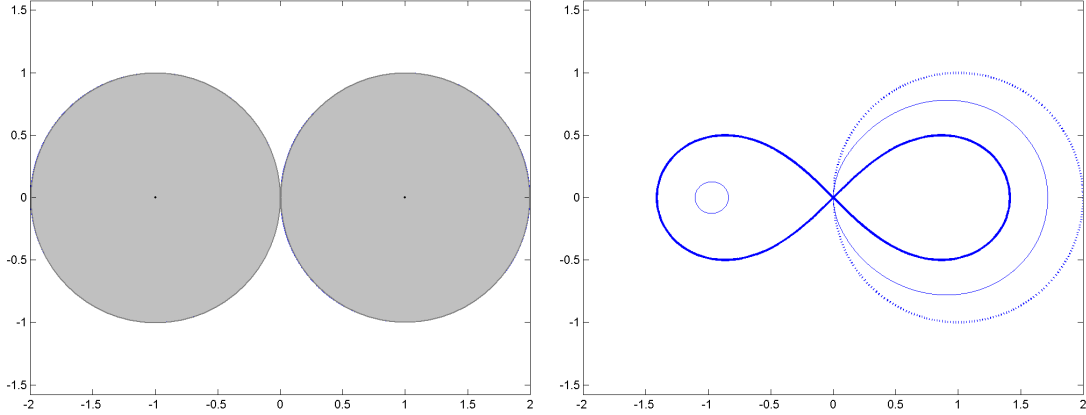


Figure 7.2: The left image shows the spectrum of the infinite random matrix (7.21). The right image shows the point spectra (solutions λ of (7.23)) corresponding to ratio $r = 0.5$ (lemniscate, bold), $r = 0.75$ (thin) and $r = 1$ (dotted).

Now take $n \in \mathbb{N}$ and $B \in \mathcal{P}_n(A)$, i.e. B is of the form (7.21), where we choose $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$ and let $\sigma_{k+n} = \sigma_k$ for all $k \in \mathbb{Z}$. Let m denote the number of 1's in $\sigma_1, \dots, \sigma_n$ so that the remaining $n - m$ entries are equal to -1 . Now we are in the situation of Case 1 ($0 \notin \mathcal{T}$) in our proof of Theorem 7.9. So take a $\lambda \in \mathbb{C}$ and look at a nontrivial solution u of $Bu = \lambda u$. Looking at (7.19) and taking into account $\tau_k = 1 \forall k$ and the periodicity of the σ_k -sequence, we get that $u \in \ell^\infty(\mathbb{Z})$ iff

$$|\lambda - 1|^m |\lambda + 1|^{n-m} = |\lambda - \sigma_1| \cdots |\lambda - \sigma_n| = 1. \quad (7.22)$$

Indeed, $|u(n)|$ from (7.19) remains bounded for $n \rightarrow +\infty$ iff the left-hand side of (7.22) is ≤ 1 , and it remains bounded for $n \rightarrow -\infty$ iff the left-hand side of (7.22) is ≥ 1 (also cf. [65]).

So we have that $\lambda \in \text{spec}_{\text{point}}^\infty B$ iff (7.22) holds. Taking n -th roots in (7.22), we get the slightly more convenient formula

$$|\lambda - 1|^r |\lambda + 1|^{1-r} = 1, \quad (7.23)$$

where $r = m/n$ is the ratio of 1's among all entries σ_k in a period of length n . The set $\text{spec}_{\text{per}} A$, as defined in (7.20), is hence equal to the set of all solutions λ of (7.23) with a rational ratio $r = m/n \in [0, 1]$.

For example, if $r = 0.5$, i.e. if n is even and $m = n/2$ then (7.23) is equivalent to $|\lambda - 1| \cdot |\lambda + 1| = 1$, which is the equation of the lemniscate with focal points -1 and 1 (see Figures 7.2 and 7.3, and cf. [175, Figures 2.1 and 3.1(b)] and [65, Figure 2]). By the same argument, it can be shown that the same lemniscate is the point spectrum not only of all periodic matrices (7.21) with an equal share of 1's and -1 's per period but also for the much larger class of all matrices of the form (7.21) for which the ratio of 1's within $\sigma_{-k}, \dots, \sigma_k$ tends to 0.5 as $k \rightarrow \infty$ – which is what one expects from a random matrix if the probability is distributed equally on $\Sigma = \{-1, 1\}$.

For $r = 0$ and $r = 1$, (7.23) is the equation of the circle with radius 1 around -1 and 1 , respectively. For every $r \in (0.5, 1)$, the solutions of (7.23) form two closed curves: one curve lies inside the left loop of the lemniscate, and the second curve lies inside the radius 1 circle around 1 but outside the right loop of the lemniscate (see the right image of Figure 7.2, also cf. the resolvent level plots in [175, Figure 2.1]).

It is easy to see that every point $\lambda \in \overline{U}_1(-1) \cup \overline{U}_1(1)$, with the only two exceptions $\lambda = -1$ and $\lambda = 1$, solves (7.23) for a particular value of $r \in [0, 1]$, namely for

$$r = \frac{1}{1 - \log_{|\lambda+1|} |\lambda-1|} \quad (7.24)$$

(the origin $\lambda = 0$, for which this formula is not applicable, is a solution of (7.23) for every $r \in [0, 1]$, and every point on the circle $|\lambda + 1| = 1$ is the solution of (7.23) for $r = 0$), and that no λ outside these two disks solves (7.23) for any value of $r \in [0, 1]$.

From (7.24) it is not hard to see that the set of all $\lambda \in \overline{U}_1(-1) \cup \overline{U}_1(1) = \text{spec} A$ for which (7.24) is rational is a dense subset of $\text{spec} A$. So here we have that the left-hand side of (7.20) is indeed dense in the right-hand side, i.e.

$$\text{spec} A = \text{clos}(\text{spec}_{\text{per}} A). \quad (7.25)$$

In this sense, for the determination of the spectrum of A , it is indeed enough to look at the periodic limit operators of A . We have tried to illustrate (7.25) in Figure 7.3.

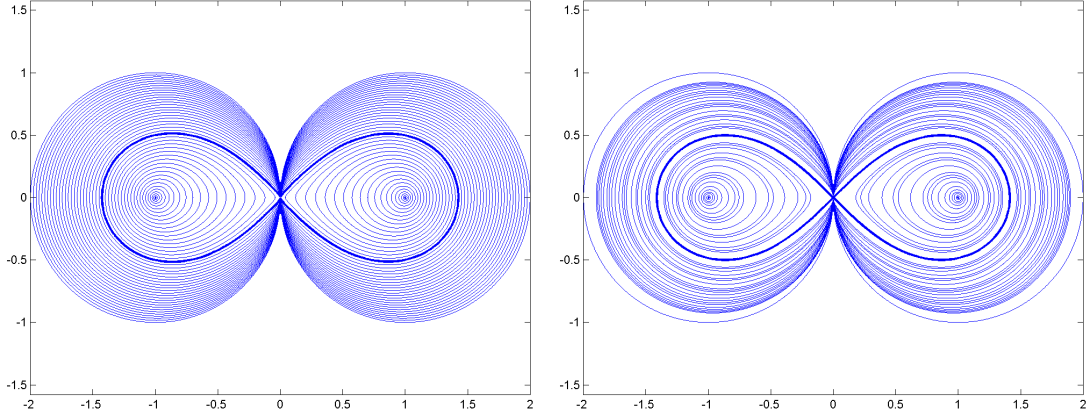


Figure 7.3: The left image shows the point spectra (solutions of (7.23)) corresponding to ratios $r = 0, 0.02, \dots, 0.98, 1$ with $r = 0.5$ highlighted (bold). So what we see on the left is $\text{spec}_{\text{per}}^{50} A$. What we see on the right is the union $\cup_{n=1}^{12} \text{spec}_{\text{per}}^n A$.

7.1.3 A Tridigonal Random Matrix

Coming from a closely related class of physical models proposed by Feinberg and Zee (see [64, 84] and the references therein) but with a much more intrinsic spectrum, we are now looking at the operator A^b that is induced by the tridiagonal matrix

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & 0 & 1 & & \\ & b_{-1} & 0 & 1 & \\ & & b_0 & 0 & 1 \\ & & & b_1 & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix} \quad (7.26)$$

with independent samples b_k from a random variable X taking values in $\Sigma = \{-1, 1\}$ and with nonzero probability for both -1 and 1 .

Again, $A^b \in BO_1(E) \subset \mathcal{W}$ acts boundedly on all spaces $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $p \in [1, \infty]$, its spectrum and essential spectrum are independent of p , and A^b is almost surely pseudoergodic whence Corollary 5.26 and (7.17) apply with

$$\sigma^{\text{op}}(A^b) = \{A^c : c \in \{\pm 1\}^{\mathbb{Z}}\}, \quad (7.27)$$

where $\{\pm 1\}^{\mathbb{Z}}$ is the set of all sequences $c = (c_k)_{k \in \mathbb{Z}}$ with all values $c_k \in \{-1, 1\}$.

The main goal of our studies is to obtain information about spectrum, pseudospectrum and numerical range of the biinfinite matrix operator A^b , its contrac-

tion A_+^b to the positive half axis (semiinfinite matrix) and its finite sections A_n^b (which are $n \times n$ submatrices of (7.26), $n \in \mathbb{N}$), i.e. the operators induced by

$$\begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & 1 & \\ & & b_3 & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & & & \\ b_1 & 0 & 1 & & \\ & b_2 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{n-1} & 0 \end{pmatrix},$$

respectively, as well as interrelations between their spectra and pseudospectra.

Note that the finite matrix A_n^b only depends on the $n - 1$ values $b_1, \dots, b_{n-1} \in \{\pm 1\}$, which is why we sometimes find it convenient to write $A_n^{b'}$ instead of A_n^b , where

$$b' := (b_1, \dots, b_{n-1}) \in \{\pm 1\}^{n-1}$$

is the corresponding finite subvector of $b = (\dots, b_{-1}, b_0, b_1, \dots) \in \{\pm 1\}^{\mathbb{Z}}$.

Because this is still very much ongoing work, together with Chandler-Wilde and Chonchaiya [31, 32], we here restrict ourselves to quoting some of the results derived so far and to giving one of the proofs that invokes formula (7.17) in a particularly illustrative way.

Theorem 7.11 *If $b \in \{\pm 1\}^{\mathbb{Z}}$ is pseudoergodic (which holds almost surely if b is random in the discussed sense) then*

a) *It holds that*

$$\text{spec } A^b = \text{spec}_{\text{ess}} A^b = \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}} \text{spec}_{\text{point}}^{\infty} A^c, \quad (7.28)$$

so in particular,

$$\text{spec}_{\text{per}} A^b := \bigcup_{n \in \mathbb{N}} \text{spec}_{\text{per}}^n A^b \subset \text{spec } A^b \quad (7.29)$$

(see Figures 7.4 and 7.5), where

$$\text{spec}_{\text{per}}^n A^b := \bigcup_{c \in \{\pm 1\}^{\mathbb{Z}}, n\text{-periodic}} \text{spec}_{\text{point}}^{\infty} A^c. \quad (7.30)$$

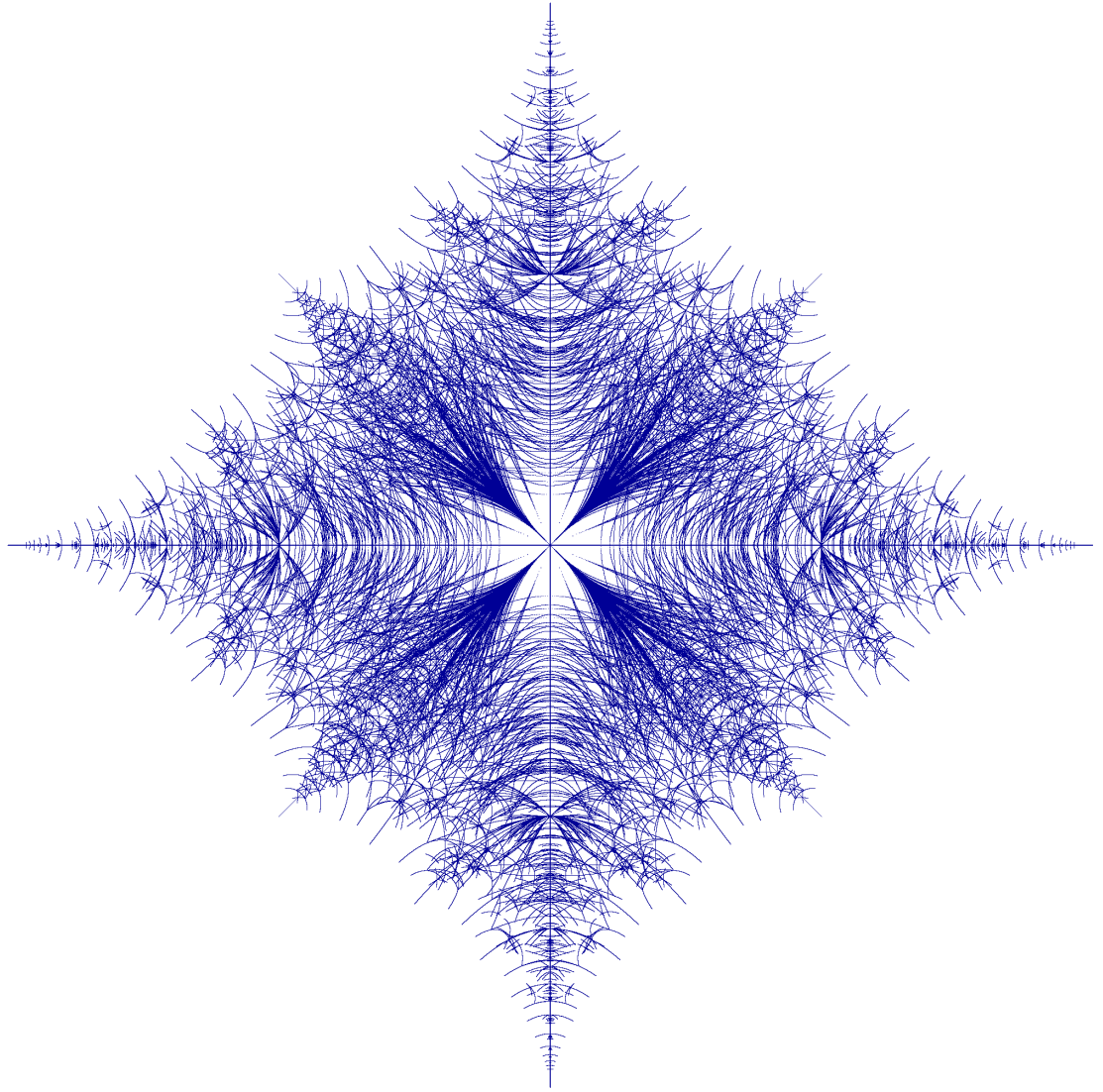


Figure 7.4: This figure shows the union $\cup_{n=1}^{12} \text{spec}_{\text{per}}^n A^b$ as defined in (7.30).

- b)** $\text{spec } A^b$ is invariant under reflection about either axis as well as under a 90° rotation around the origin.
- c)** For all $\varepsilon \geq 0$ and $p \in [1, \infty]$, one has

$$\text{spec}_\varepsilon^p A^b = \text{spec}_\varepsilon^p A_+^b.$$

d) *It holds that*

$$\operatorname{spec} A^b \subset \overline{\operatorname{Num} A^b} = \operatorname{conv}\{2, -2, 2i, -2i\},$$

where $\overline{\operatorname{Num} A^b}$ denotes the closure of the numerical range of A^b .

e) $\operatorname{spec} A^b$ contains the closed unit disk.

f) For every $n \in \mathbb{N}$, it holds that

$$\operatorname{spec} A_n^b \subset \bigcup_{c \in \{\pm 1\}^{n-1}} \operatorname{spec} A_n^c \subset \operatorname{spec}_{\operatorname{per}}^{2n+2} A^b \subset \operatorname{spec} A^b \quad (7.31)$$

(see Figures 7.6 – 7.11), where $\operatorname{spec}_{\operatorname{per}}^m A^b$ is as defined in (7.30).

g) For all $n \in \mathbb{N}$, $\varepsilon \geq 0$ and $p \in [1, \infty]$, one has

$$\operatorname{spec}_{\varepsilon}^p A_n^b \subset \bigcup_{c \in \{\pm 1\}^{n-1}} \operatorname{spec}_{\varepsilon}^p A_n^c \subset \operatorname{spec}_{\varepsilon}^p A^b.$$

Currently, in an approach that applies to a large range of band operators [31], we are about to complement f) and g) by an upper bound on $\operatorname{spec}_{\varepsilon}^p A^b$ of the form

$$\operatorname{spec}_{\varepsilon}^p A^b \subset \bigcup_{c \in \{\pm 1\}^{n-1}} \operatorname{spec}_{\varepsilon + f_p(n)}^p A_n^c$$

for $n \in \mathbb{N}$, $\varepsilon \geq 0$ and $p \in [1, \infty]$, where the function $f_p(n)$ is explicitly known and goes to zero as $n \rightarrow \infty$. Together with g) this proves (and quantifies) the convergence of the (pseudo)spectra $\operatorname{spec}_{\varepsilon}^p A_n^b$ to $\operatorname{spec}_{\varepsilon}^p A^b$ as $n \rightarrow \infty$, for all $\varepsilon \geq 0$.

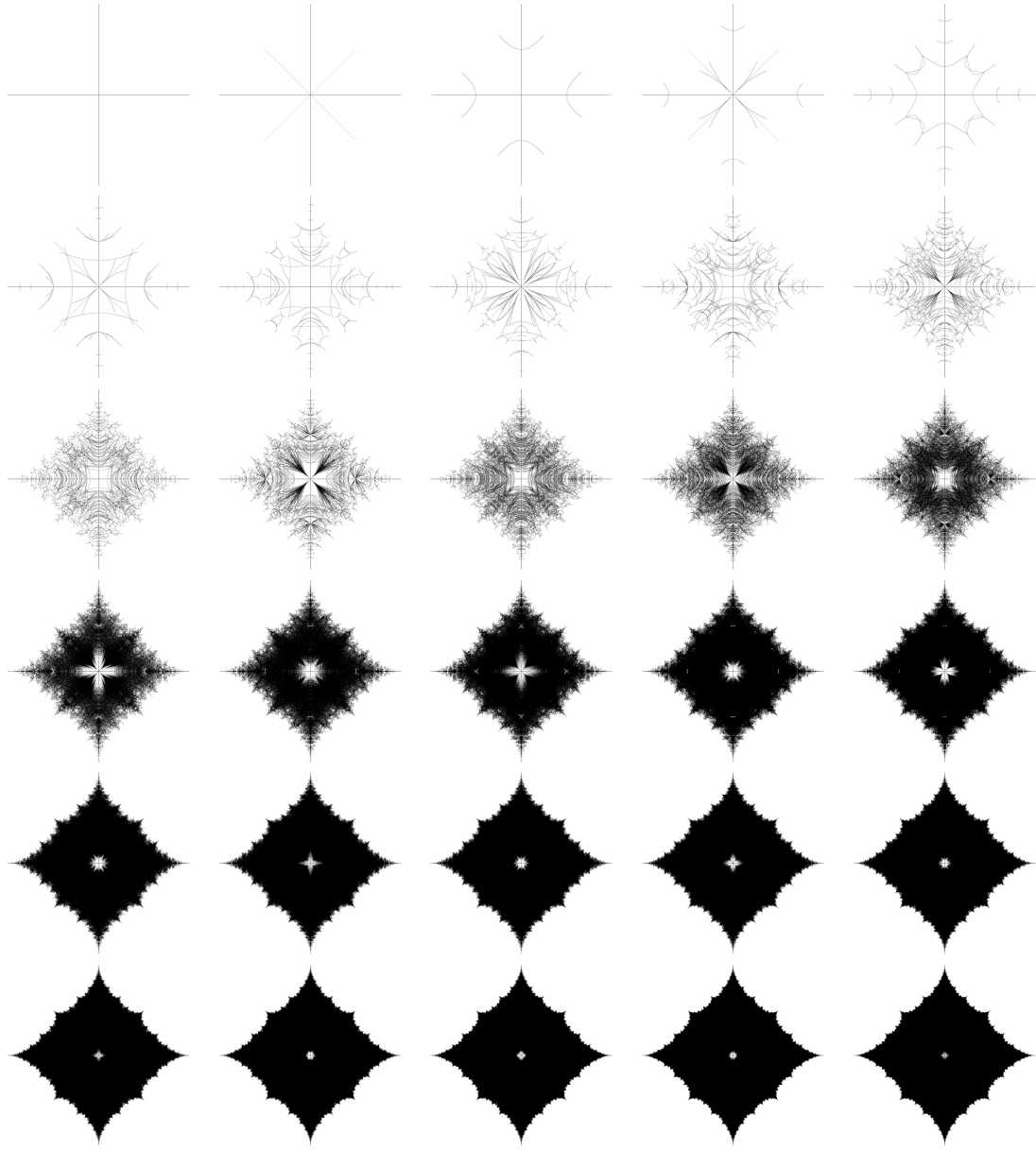


Figure 7.5: Our figure shows the sets $\text{spec}_{\text{per}}^n A^b$, as defined in (7.30), for $n = 1, \dots, 30$. Note that each set $\text{spec}_{\text{per}}^n A^b$ consists of k analytic arcs, where $2^n/n \leq k \leq 2^n$. Recall that Figure 7.4 shows the union of the first twelve pictures of this figure.

From Theorem 7.11 a) and f) we know that all the sets in Figures 7.5 and 7.6 are contained in $\text{spec } A^b$. We conjecture that, similarly to the previous section, the left-hand side of (7.29) is dense in the right-hand side, $\text{spec } A^b$.

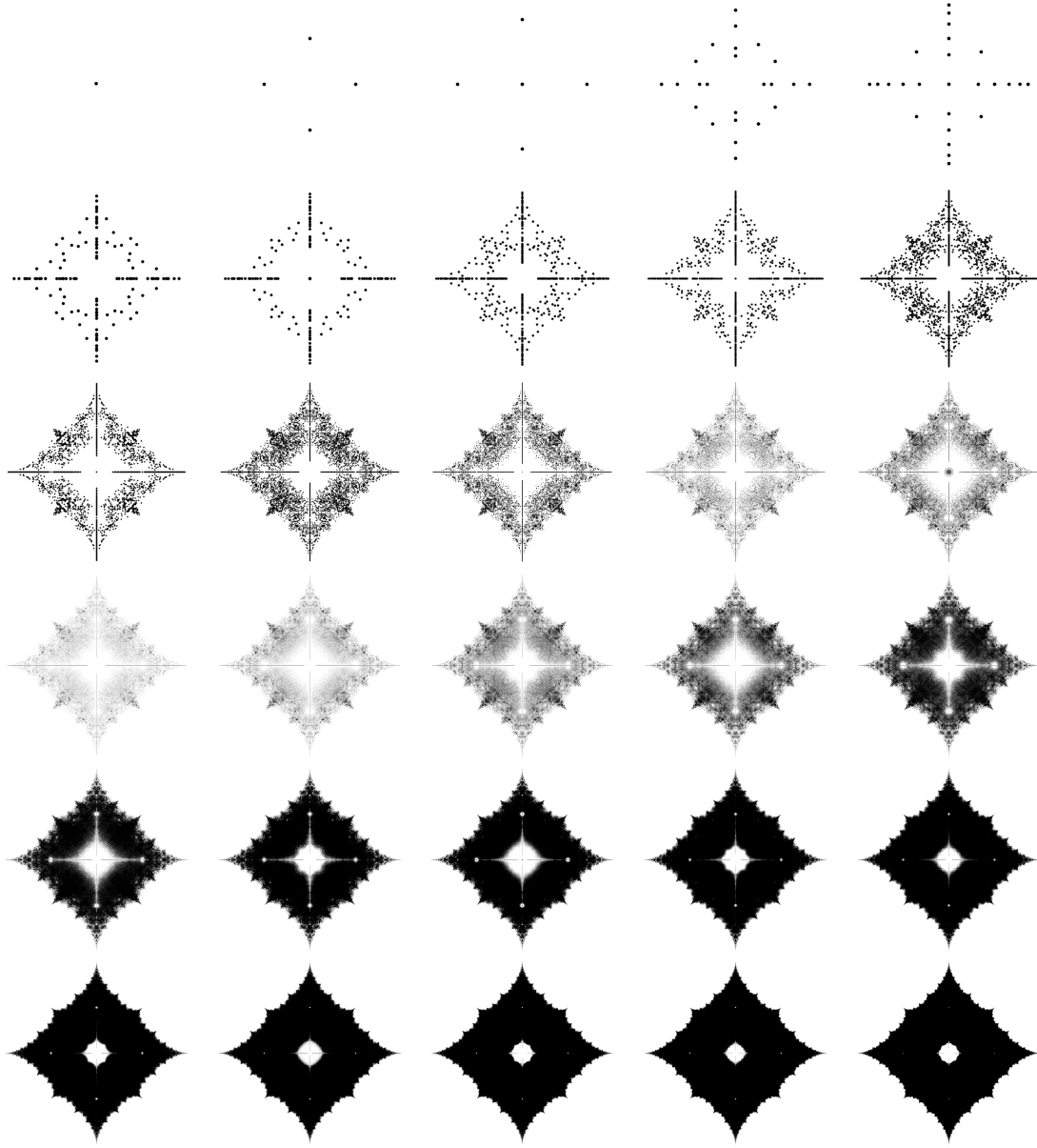


Figure 7.6: Our figure shows the unions $\cup_{c \in \{\pm 1\}^{n-1}} \text{spec } A_n^c$ of all $n \times n$ matrix eigenvalues for $n = 1, \dots, 30$. Note that in the first pictures (with only few eigenvalues), we have used heavier pixels for the sake of visibility. By (7.31), each of the sets with $n = 1, 2, \dots, 14$ in this figure is contained, respectively, in the set number $2n + 2$ of Figure 7.5. We illustrate this for the particular cases $n = 4, 5, 9, 10$ in Figures 7.8 – 7.11.

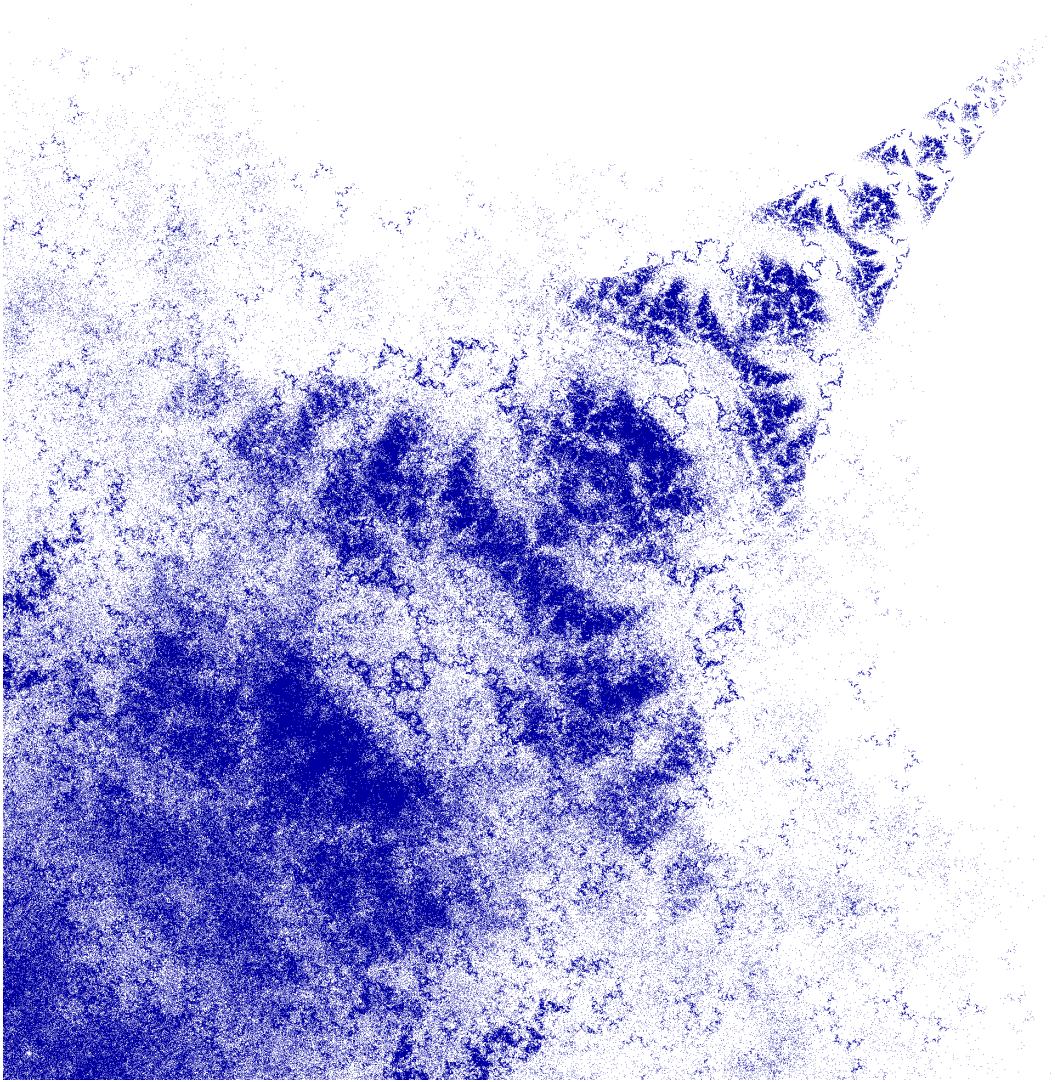


Figure 7.7: This is a zoom into $\cup_{c \in \{\pm 1\}^{24}} \text{spec } A_{25}^c$, i.e. the 25th picture of Figure 7.6.

Proof of Theorem 7.11 e)

Fix $\lambda \in \mathbb{C}$. Formula (7.28) for the spectrum of A^b with a pseudoergodic $b \in \{\pm 1\}^{\mathbb{Z}}$ motivates the following approach: We are looking for a sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ such that $\lambda \in \text{spec}_{\text{point}}^{\infty} A^c$; that is, there exists an $u \in \ell^{\infty}(\mathbb{Z})$ with $A^c u = \lambda u$, i.e.

$$u(i+1) = \lambda u(i) - c_i u(i-1) \quad (7.32)$$

for every $i \in \mathbb{Z}$. If such a sequence c exists then $\lambda \in \text{spec } A^b$ – if not then not.

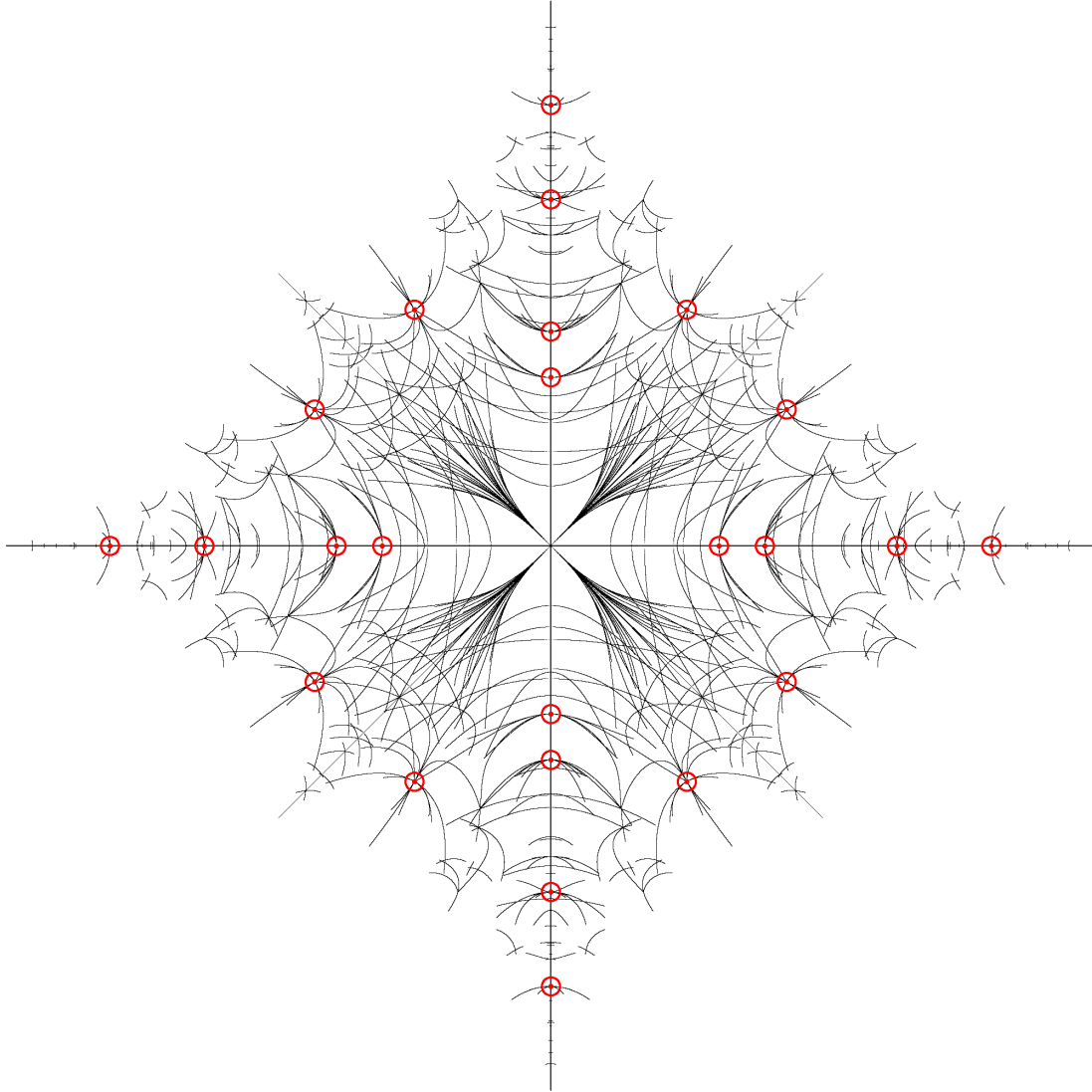


Figure 7.8: Here we see the set $\cup_{c \in \{\pm 1\}^3} \text{spec } A_4^c$ of 4×4 eigenvalues (circled) as a subset of $\text{spec}_{\text{per}}^{10} A^b$, which holds by (7.31) with $n = 4$.

Starting from $u(0) = 0$ and $u(1) = 1$, we will successively use (7.32) to compute $u(i)$ for $i = 2, 3, \dots$ (an analogous procedure is possible for $i = -1, -2, -3, \dots$). Doing so we get

$$\begin{aligned} u(2) &= \lambda, & u(3) &= \lambda^2 - c_2, & u(4) &= \lambda^3 - (c_2 + c_3)\lambda, \\ u(5) &= \lambda^4 - (c_2 + c_3 + c_4)\lambda^2 + c_2c_4, \dots \end{aligned}$$

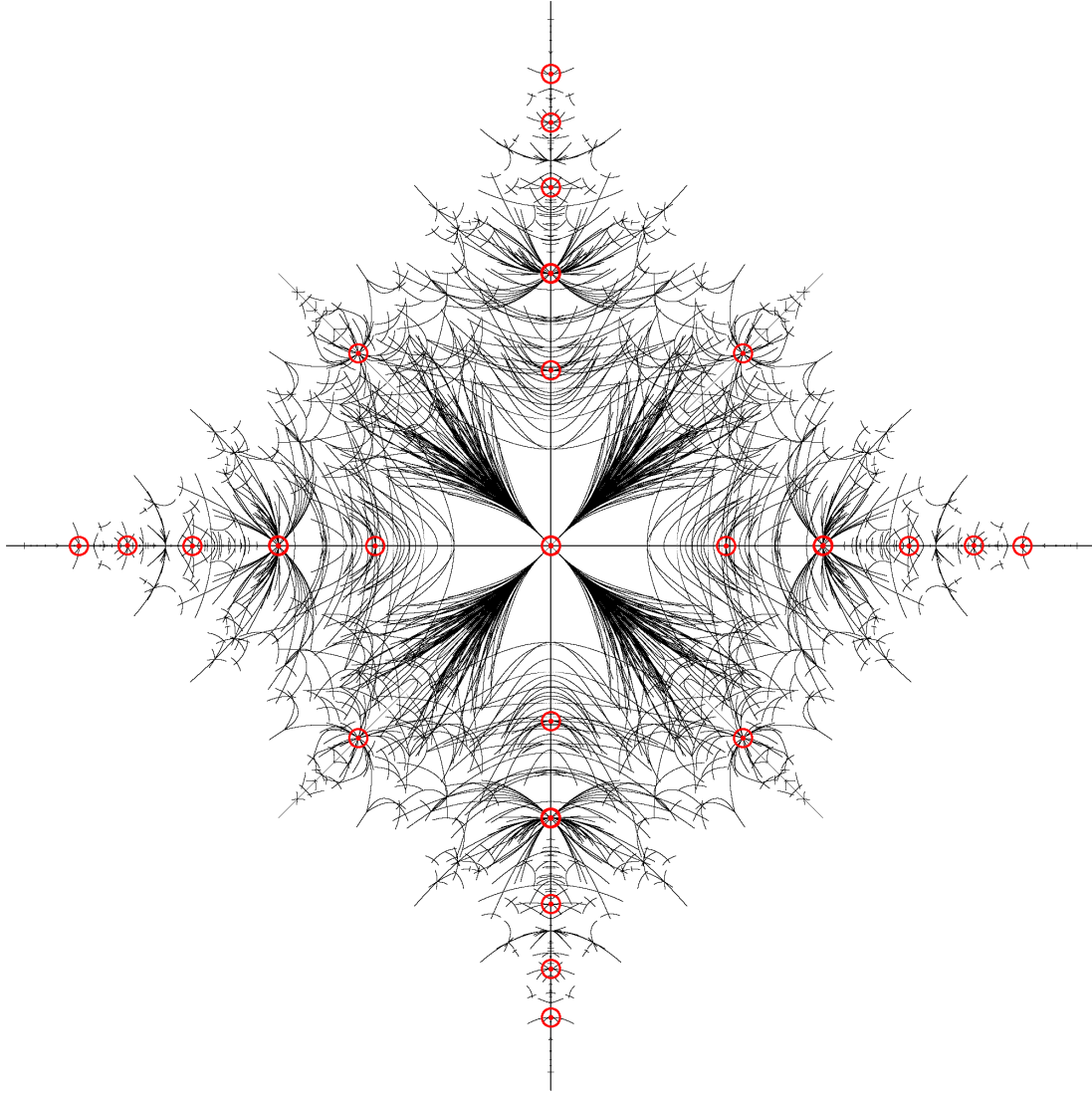


Figure 7.9: Here we see the set $\cup_{c \in \{\pm 1\}^4} \text{spec } A_5^c$ of 5×5 eigenvalues (circled) as a subset of $\text{spec}_{\text{per}}^{12} A^b$, which holds by (7.31) with $n = 5$.

Remark 7.12 It is easy to check that the solution of (7.32) with initial conditions $u(0) = 0$, $u(1) = 1$ and $u(2) = \lambda$ coincides with the characteristic polynomial

$$u(i) = \begin{vmatrix} \lambda & -1 & & & \\ -c_2 & \lambda & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -c_{i-1} & \lambda & \end{vmatrix}, \quad i = 3, 4, \dots$$

□

In general, $u(i)$ is a polynomial of degree $i - 1$ in λ with coefficients depending on c_2, \dots, c_{i-1} . Since we want u to be a bounded sequence, we are trying to keep the coefficients of these polynomials small. Precisely, our strategy will be to choose $c_1, c_2, \dots \in \{\pm 1\}$ such that each $u(i)$ is a polynomial in λ with coefficients in $\{-1, 0, 1\}$. The following table, where we abbreviate -1 by $-$, $+1$ by $+$, and 0 by a space, shows that this seems to be possible.

i	c_i	$j \rightarrow$	coefficients of λ^{j-1} in the polynomial $u(i)$															
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	\dots
1	+	+																
2	+		+															
3	-	-		+														
4	-				+													
5	+	-		+		+												
6	-		-				+											
7	+	-				+		+										
8	-								+									
9	+	-				+		+		+								
10	+		-				+				+							
11	-	+		-		-				-		+						
12	+				-								+					
13	-	-		+						+		-		+				
14	-		-								+				+			
15	+	-								+				+		+		
16	-																+	
\vdots	\vdots									\vdots								

(7.33)

(7.33)

For $i, j \in \mathbb{N}$, we denote the coefficient of λ^{j-1} in the polynomial $u(i)$ by $p(i, j)$. Then the right part of table (7.33) shows the values $p(i, j)$ for $i, j = 1, \dots, 16$. From (7.32) it follows that

$$p(i+1, j) = p(i, j-1) - c_i p(i-1, j) \quad (7.34)$$

holds for $i = 2, 3, \dots$ and $j = 1, 2, \dots, i$ with $p(i', j') := 0$ if $j' < 1$ or $j' > i'$.

If, for some i, j , one has that $p(i, j-1) \neq 0$ and $p(i-1, j) \neq 0$ then, by (7.34) and $p(i, j-1), p(i-1, j) \in \{-1, 1\}$, this implies that $p(i+1, j) = 0$, i.e.

$$c_i = p(i, j-1)/p(i-1, j) = p(i, j-1) \cdot p(i-1, j) \quad (7.35)$$

since otherwise $p(i+1, j) \in \{-2, 2\}$. As an example, look at $p(15, 1) = -1$ and $p(14, 2) = -1$. If $c_{15} = -1$, we would get from (7.34) that $p(16, 2) = -2 \notin \{-1, 0, 1\}$, so it remains to take $c_{15} = 1 = p(15, 1) \cdot p(14, 2)$. The same value $c_{15} = 1$ is enforced by $p(15, 9)$ and $p(14, 10)$, as well as by $p(15, 13)$ and $p(14, 14)$.

We will prove that this coincidence, i.e. that the right-hand side of (7.35) is (if non-zero) independent of j , is not a matter of fortune. As a result we get that the table (7.33) continues without end, only using values from $\{-1, 0, 1\}$ for $p(i, j)$ and from $\{\pm 1\}$ for c_i . To prove this, we employ a particular self-similarity in the triangular pattern of (7.33); more precisely, it can be shown that the pattern of non-zero entries of $p(\cdot, \cdot)$ forms a so-called infinite discrete Sierpinski triangle.

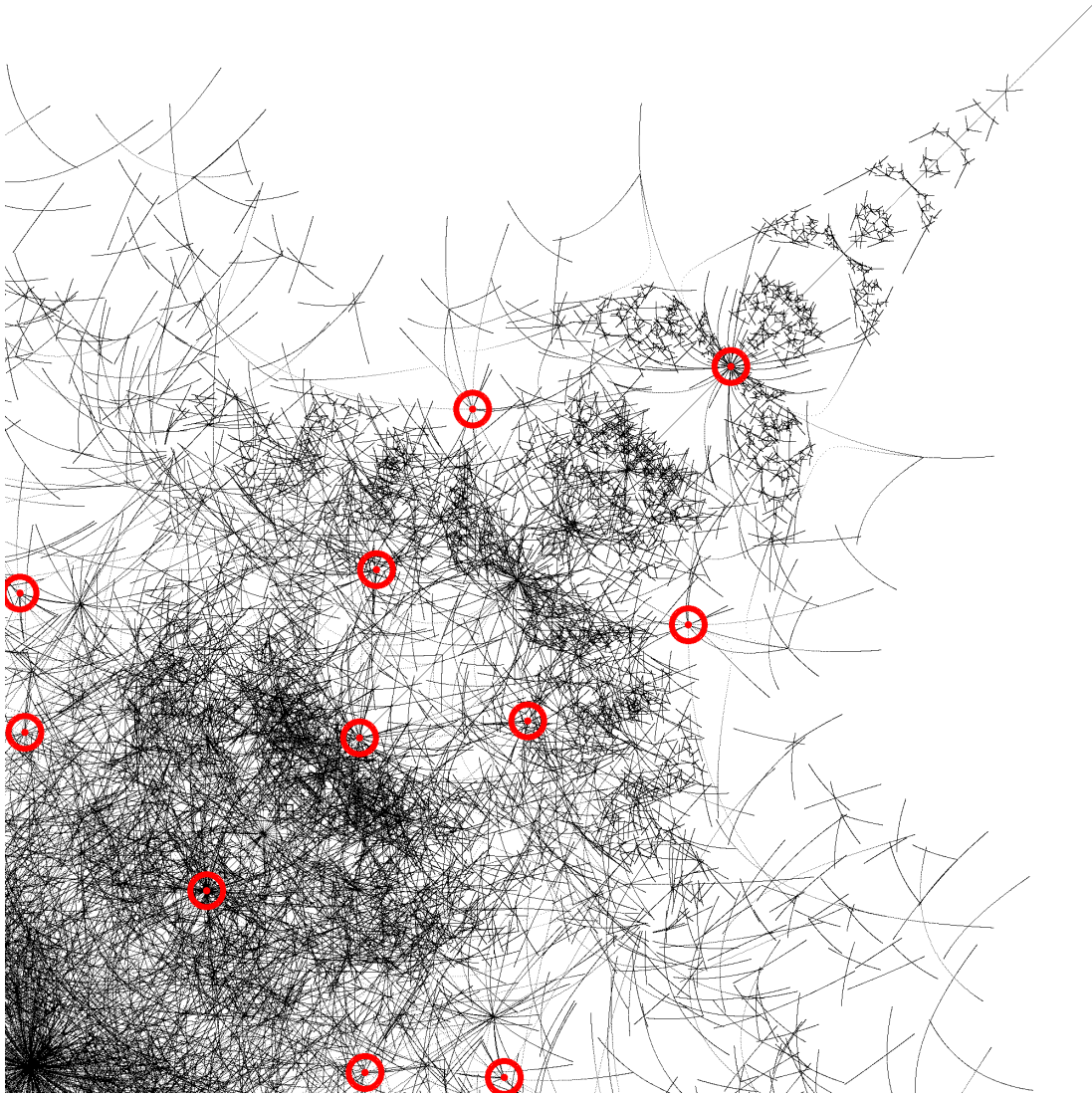


Figure 7.10: This is a zoom into the set $\cup_{c \in \{\pm 1\}} \text{spec } A_9^c$ of 9×9 matrix eigenvalues (circled) as a subset of $\text{spec}_{\text{per}}^{20} A^b$, which holds by (7.31) with $n = 9$.

Proposition 7.13 Define the sequence $c \in \{\pm 1\}^{\mathbb{Z}}$, for positive indices by $c_1 = 1$ and by the requirement that

$$c_{2i} = c_{2i-1} c_i \quad \text{and} \quad c_{2i+1} = -c_{2i}, \quad i = 1, 2, \dots,$$

and for negative indices by

$$c_{-i} = c_{i+1}, \quad i = 0, 1, \dots$$

Further, given $\lambda \in \mathbb{C}$, define the sequence $u = (u(i))_{i \in \mathbb{Z}}$, by the requirement that

$$u(i+1) = \lambda u(i) - c_i u(i-1), \quad i \in \mathbb{Z},$$

and by the initial conditions

$$u(0) = 0, \quad u(1) = 1.$$

Then, as a function of λ , for $i \in \mathbb{Z}$, $u(i)$ is a polynomial of degree $|i| - 1$ with all its coefficients taking values in the set $\{-1, 0, 1\}$. More precisely, for every $i \in \mathbb{N}$,

$$u(i) = \sum_{j=1}^i p(i, j) \lambda^{j-1},$$

where $p(1, 1) = 1$ and

(i) it holds that

$$\begin{pmatrix} p(2i-1, 2j-1) & p(2i-1, 2j) \\ p(2i, 2j-1) & p(2i, 2j) \end{pmatrix} = \begin{cases} p(i, j) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } i+j \text{ is even,} \\ c_{2i-1} p(i-1, j) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } i+j \text{ is odd} \end{cases}$$

for every $j = 1, \dots, i$, and

(ii) $p(i, j-1) \cdot p(i-1, j) \in \{0, c_i\}$ for all $j = 2, \dots, i-1$.

So in particular, by (i), all values $p(i, j)$ are in $\{-1, 0, 1\}$. Further, for $i \in \mathbb{N}$,

$$u(-i) = d_i u(i),$$

where, for $i \in \mathbb{N}$,

$$d_{2i-1} = (-1)^i, \quad d_{2i} = (-1)^i c_{2i}.$$

Remark 7.14 Statement (i) reveals the self-similar structure of the pattern (7.33): An entry $p(i, j)$ replicates three times – as $p(2i-1, 2j-1)$, $p(2i, 2j)$ and, multiplied by c_{2i+1} , as $p(2i+1, 2j-1)$. So, if scaled by a factor 2, the “volume” of the pattern triples, which is why its fractal dimension is $\log_2 3 \approx 1.585$ – exactly as for its bounded version, the usual Sierpinski triangle. \square

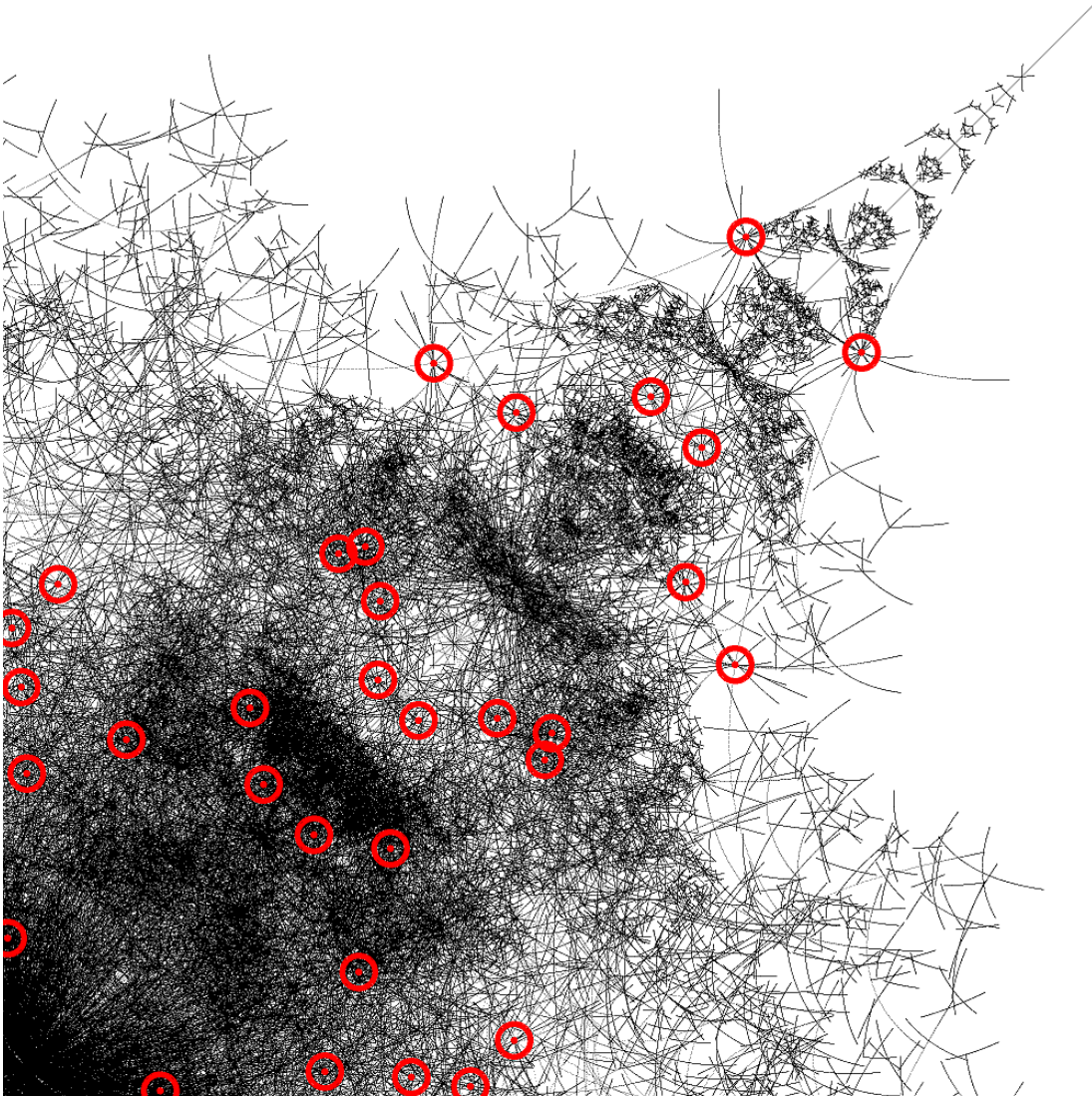


Figure 7.11: This is a zoom into the set $\cup_{c \in \{\pm 1\}^{\mathbb{Z}}} \text{spec } A_{10}^c$ of 10×10 matrix eigenvalues (circled) as a subset of $\text{spec}_{\text{per}}^{22} A^b$, which holds by (7.31) with $n = 10$.

As an immediate consequence of Proposition 7.13 we get the following result.

Corollary 7.15 *For the sequence $c \in \{\pm 1\}^{\mathbb{Z}}$ from Proposition 7.13, it holds that the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ is contained in $\text{spec } A^c$. Consequently, for every pseudoergodic $b \in \{\pm 1\}^{\mathbb{Z}}$, one has $\overline{\mathbb{D}} \subset \text{spec } A^b$.*

Proof. Let $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, let c be the sequence from Proposition 7.13 and $u : \mathbb{Z} \rightarrow \mathbb{C}$ the eigenfunction from (7.32). Then, for every $i \in \mathbb{Z}$,

$$|u(i)| = \left| \sum_{j=1}^{|i|} p(i, j) \lambda^{j-1} \right| \leq \sum_{j=1}^{|i|} |p(i, j)| |\lambda|^{j-1} \leq \sum_{j=1}^{\infty} |\lambda|^{j-1} = \frac{1}{1 - |\lambda|},$$

showing that $u \in \ell^\infty(\mathbb{Z})$, and, by our construction (7.32), $A^c u = \lambda u$. So $\mathbb{D} \subset \text{spec}_{\text{point}}^\infty A^c \subset \text{spec } A^c$. Since $\text{spec } A^c$ is closed, it holds that $\overline{\mathbb{D}} \subset \text{spec } A^c$. The claim for a pseudoergodic b then follows from $\text{spec } A^c \subset \text{spec } A^b$ since A^c is a limit operator of A^b . ■

Proof of Proposition 7.13. Firstly, it is easy to see (by (7.33), (7.34) and induction) that $p(i', j') = 0$ if $i' + j'$ is odd, whence $p(2i - 1, 2j) = 0 = p(2i, 2j - 1)$ for all i, j .

We will now prove (i) and (ii) by induction over $i \in \mathbb{N}$. Therefore, let (i) be satisfied for $i = 1, \dots, k$, and let (ii) be satisfied for $i = 1, \dots, 2k$. (The base case is easily verified by looking at table (7.33)). We will then prove (i) for $i = k + 1$ and (ii) for $i = 2k + 1$ and $2k + 2$.

Part (i). We let $i = k + 1$ and start with the case when $i + j$ is even. By (7.34), we have that

$$\begin{aligned} p(2i - 1, 2j - 1) &= p(2i - 2, 2j - 2) - c_{2i-2} \cdot p(2i - 3, 2j - 1) \\ &= p(2(i - 1), 2(j - 1)) - c_{2i-2} \cdot p(2(i - 1) - 1, 2j - 1), \end{aligned} \quad (7.36)$$

where, by induction (and since $i - 1 + j - 1$ is even), $p(2(i - 1), 2(j - 1)) = p(i - 1, j - 1)$ if $j > 1$ and it is 0 if $j = 1$. Also by induction, $p(2(i - 1) - 1, 2j - 1) = c_{2(i-1)-1} p(i - 2, j)$ since $i - 1 + j$ is odd. To determine c_{2i-2} , take $J \in \{1, \dots, 2i - 4\}$ such that $p(2i - 2, J) \neq 0$ (whence $J =: 2j'$ has to be even) and $p(2i - 3, J + 1) \neq 0$ (if no such J exists then we are free to choose c_{2i-2} in which case we will put $c_{2i-2} := c_{2i-3} c_{i-1}$). From (i) and $0 \neq p(2i - 2, J) = p(2(i - 1), 2j')$ it is clear that $i - 1 + j'$ is even and $i - 1 + j' + 1$ is odd. Now, by (ii) and (i), we have that

$$\begin{aligned} c_{2i-2} &= p(2i - 2, J) p(2i - 3, J + 1) \\ &= p(2(i - 1), 2j') p(2(i - 1) - 1, 2(j' + 1) - 1) \\ &= p(i - 1, j') c_{2(i-1)-1} p(i - 2, j' + 1) = c_{2i-3} c_{i-1}. \end{aligned}$$

Inserting all these results in (7.36), we get that

$$\begin{aligned} p(2i - 1, 2j - 1) &= \begin{cases} p(i - 1, j - 1) - c_{2i-3} c_{i-1} c_{2i-3} p(i - 2, j) & \text{if } j > 1, \\ 0 - c_{2i-3} c_{i-1} c_{2i-3} p(i - 2, j) & \text{if } j = 1 \end{cases} \\ &= \begin{cases} p(i - 1, j - 1) - c_{i-1} p(i - 2, j) & \text{if } j > 1, \\ -c_{i-1} p(i - 2, j) & \text{if } j = 1 \end{cases} = p(i, j). \end{aligned}$$

We saw that $p(2i-1, 2j) = 0 = p(2i, 2j-1)$ for all i, j ; so we are left with

$$p(2i, 2j) = p(2i-1, 2j-1) - c_{2i-1}p(2i-2, 2j) = p(i, j) - c_{2i-1}0 = p(i, j).$$

Now suppose $i+j$ is odd. Then, almost exactly as above,

$$\begin{aligned} p(2i-1, 2j-1) &= p(2i-2, 2j-2) - c_{2i-2}p(2i-3, 2j-1) \\ &= 0 - c_{2i-3}c_{i-1}p(i-1, j) = c_{2i-1}p(i-1, j) \end{aligned}$$

with $c_{2i-1} := -c_{2i-2} = -c_{2i-3}c_{i-1}$, and

$$\begin{aligned} p(2i, 2j) &= p(2i-1, 2j-1) - c_{2i-1}p(2i-2, 2j) \\ &= c_{2i-1}p(i-1, j) - c_{2i-1}p(i-1, j) = 0 \end{aligned}$$

since $i-1+j$ is even.

Part (ii). Let $i = 2k+1$ and suppose $j \in \{2, \dots, 2k\}$ is such that $p(i, j-1) \neq 0$ (whence $i+j-1$ is even, i.e. $j =: 2j'$ is even) and $p(i-1, j) \neq 0$. If no such j exists then the product in (ii) is always zero and there is nothing to show. From $0 \neq p(i-1, j) = p(2k, 2j')$ and (i) we get that $k+j'$ is even and $k+1+j'$ is odd. Now we have that

$$\begin{aligned} p(i, j-1)p(i-1, j) &= p(2k+1, 2j'-1)p(2k, 2j') \\ &= p(2(k+1)-1, 2j'-1)p(2k, 2j') \\ &= c_{2(k+1)-1}p(k, j')p(k, j') = c_{2k+1} = c_i \end{aligned}$$

is independent of j . Now let $i = 2k+2$ and suppose $j \in \{2, \dots, 2k+1\}$ is such that $p(i, j-1) \neq 0$ (whence $i+j-1$ is even, i.e. $j =: 2j'+1$ is odd) and $p(i-1, j) \neq 0$. (Again, if no such j exists then there is nothing to show.) From $0 \neq p(i, j-1) = p(2k+2, 2j') = p(2(k+1), 2j')$ and (i) we get that $k+1+j'$ is even and $k+1+j'+1$ is odd. Now we have that

$$\begin{aligned} p(i, j-1)p(i-1, j) &= p(2k+2, 2j')p(2k+1, 2j'+1) \\ &= p(2(k+1), 2j')p(2(k+1)-1, 2(j'+1)-1) \\ &= p(k+1, j')c_{2(k+1)-1}p(k, j'+1) \\ &= c_{2k+1}c_{k+1} = c_{2k+2} = c_i \end{aligned}$$

is independent of j . ■

Remark 7.16 Note that the sequence c in Proposition 7.13 is not pseudoergodic since, by $c_{2i+1} = -c_{2i}$, the patterns “+ + +” and “- - -” can never occur as consecutive entries in the sequence c . □

7.1.4 A Class of Integral Operators

In this chapter, we apply the results of Chapter 5 to study a class of operators on

$$E = E^p(X) = \begin{cases} \ell^p(\mathbb{Z}^N, X), & p \in [1, \infty], \\ c_0(\mathbb{Z}^N, X), & p = 0 \end{cases}$$

with $X = L^q([0, 1]^N)$ for some fixed $q \in (1, \infty]$. In the natural way, as introduced in (2.4), we identify elements $u \in E$ with equivalence classes of scalar-valued functions on \mathbb{R}^N and denote the set of all of these (equivalence classes of) functions f with

$$\|f\|_{p,q} := \|u\|_E < \infty$$

by $L^{p,q}(\mathbb{R}^N)$ or just $L^{p,q}$. Note that $L^{q,q}(\mathbb{R}^N) = L^q(\mathbb{R}^N)$. Equipped with the norm $\|\cdot\|_{p,q}$, $L^{p,q}$ is a Banach space, and (2.4) yields an isometric isomorphism between $L^{p,q}$ and $E^p(X)$. We will freely identify these two spaces and the notions of strict convergence, limit operators, as well as the operators $P_m, V_k \in L(E^p(X))$ with the corresponding notions and operators on $L^{p,q}$ (cf. [106, (1.3)]).

The operators we are going to study on E alias $L^{p,q}$ are assembled, via addition and composition, from two basic ingredients:

- For $b \in L^\infty := L^\infty(\mathbb{R}^N)$, define the *multiplication operator* $\mathbf{M}_b \in L(L^{p,q})$ by

$$(\mathbf{M}_b f)(x) = b(x) f(x), \quad x \in \mathbb{R}^N$$

for all $f \in L^{p,q}$. Via the identification (2.4) between $L^{p,q}$ and E , we can identify \mathbf{M}_b with the generalised multiplication operator M_c on E from Definition 3.41, where $c \in \ell^\infty(\mathbb{Z}^N, L(X))$ is such that $(c(m)u)(x) = b(m+t)u(x)$ for all $m \in \mathbb{Z}^N$, $u \in X = L^q([0, 1]^N)$ and $x \in [0, 1]^N$. Recall from Example 5.36 that \mathbf{M}_b is rich iff $\{b(\cdot + k)\}_{k \in \mathbb{Z}^N}$ is relatively sequentially compact in the strict topology on L^∞ , in which case we write $b \in L^\infty_\mathfrak{s}$. It is easy to check that $L^\infty_\mathfrak{s}$ is an inverse closed Banach subalgebra of L^∞ .

- For $\kappa \in L^1 := L^1(\mathbb{R}^N)$, define the *convolution operator* $\mathbf{C}_\kappa \in L(L^{p,q})$ by

$$(\mathbf{C}_\kappa f)(x) = (\kappa \star f)(x) = \int_{\mathbb{R}^N} \kappa(x - y) f(y) dy, \quad x \in \mathbb{R}^N$$

for all $f \in L^{p,q}$. As demonstrated in [106, Example 1.28], the convolution operator \mathbf{C}_κ on $L^{p,q}$ corresponds to a Laurent operator L on E , where every entry λ_k of $[L] = (\lambda_{i-j})_{i,j \in \mathbb{Z}^N}$ is the operator of convolution by $\kappa(\cdot + k)$ on X . By Young's inequality [147], we get that

$$\|\lambda_k\| \leq \|\kappa|_{k+[-1,1]^N}\|_1$$

for every $k \in \mathbb{Z}^N$, and by $\kappa \in L^1(\mathbb{R}^N)$ it follows that $L \in \mathcal{W} = \mathcal{W}(X)$.

We know that \mathbf{M}_b is rich iff $b \in L^\infty_{\mathfrak{S}}$, i.e. if every sequence in \mathbb{Z}^N tending to infinity has an infinite subsequence $h = (h_m)$ such that there exists a function $c \in L^\infty$, that is later denoted by $b^{(h)}$, with

$$\|b|_{h_m+U} - c|_U\|_\infty \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad (7.37)$$

for every compact set $U \subset \mathbb{R}^N$. A straightforward computation (see Section 3.4.11 of [106] or [105] for much more on this) shows that the operator $\mathbf{C}_\kappa \mathbf{M}_b$ with $\kappa \in L^1$ is rich as an operator on L^∞ if the above holds with (7.37) replaced by the much weaker condition

$$\|b|_{h_m+U} - c|_U\|_1 \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (7.38)$$

We denote the set of all $b \in L^\infty$ with this property by $L^\infty_{\text{SC}\mathfrak{S}}$ and write $\tilde{b}^{(h)}$ for the function c with property (7.38) for all compact sets U .

We denote by \mathcal{A}^o the smallest algebra in $L(L^{p,q})$ containing all operators of these two types; that is the set of all finite sum-products of operators of the form \mathbf{M}_b and \mathbf{C}_κ with $b \in L^\infty_{\mathfrak{S}}$ and $\kappa \in L^1$. From the above considerations it follows that every operator $A \in \mathcal{A}^o$, if identified with an operator on E , is contained in the Wiener algebra \mathcal{W} . By \mathcal{A} we denote the closure of \mathcal{A}^o in the norm $\|\cdot\|_{\mathcal{W}}$. Note that, by (3.22), the closure of a set $S \subset \mathcal{W}$ in the \mathcal{W} -norm is always contained in the closure of S in the usual operator norm.

Lemma 7.17 *The predual space X^\triangleleft exists and, if $p = \infty$, then every $A \in \mathcal{A}$ has a preadjoint operator A^\triangleleft on $L^{1,q'}$ with $1/q + 1/q' = 1$.*

Proof. By the choice $q \in (1, \infty]$, it is clear that the predual space X^\triangleleft of $X = L^q([0, 1]^N)$ exists and can be identified with $L^{q'}([0, 1]^N)$, where $1/q + 1/q' = 1$, including the case $q' = 1$ if $q = \infty$. Now suppose $p = \infty$. Then the predual $E^1(X^\triangleleft)$ of $E = E^\infty(X)$ exists and corresponds to $L^{1,q'}$ in the sense of (2.4). By Proposition 3.30 and the fact that both multiplication and convolution operators have a preadjoint operator (indeed, $\mathbf{M}_b^\triangleleft = \mathbf{M}_b$ and $\mathbf{C}_\kappa^\triangleleft = \mathbf{C}_{\kappa(-\cdot)}$ for all $b \in L^\infty$ and $\kappa \in L^1$), we see that indeed A^\triangleleft exists for every $A \in \mathcal{A}$. ■

Now let

$$\mathcal{J}^o := \left\{ \sum_i A_i \mathbf{C}_{\kappa_i} B_i : A_i, B_i \in \mathcal{A}^o, \kappa_i \in L^1 \right\},$$

with the sum being finite, denote the smallest two-sided ideal of \mathcal{A}^o containing all convolution operators \mathbf{C}_κ with $\kappa \in L^1$, and let \mathcal{J} be its closure in the norm $\|\cdot\|_{\mathcal{W}}$, hence the smallest \mathcal{W} -closed two-sided ideal of \mathcal{A} containing all \mathbf{C}_κ .

Lemma 7.18 *It holds that $\mathcal{J} \subset UM(L^{p,q})$. In particular, every operator in \mathcal{J} is Montel.*

Proof. The inclusion $\mathcal{J} \subset UM(L^{p,q})$ follows from Lemma 5.4 (a),(c),(d) and the fact that $C_\kappa \in UM(L^{p,q})$ for all $\kappa \in L^1$ since the set $\{V_{-k}KV_k : k \in \mathbb{Z}^N\}$ in Lemma 5.2 (ii) is just a singleton if $K = C_\kappa$. ■

It can be shown that, in the same way as every $A \in \mathcal{A}^o$ clearly can be written as the sum of a multiplication operator and an operator in \mathcal{J}^o , also every $A \in \mathcal{A}$ can be uniquely written as

$$A = M_b + K \quad \text{with} \quad b \in L^\infty_{\mathbb{S}} \quad \text{and} \quad K \in \mathcal{J}. \quad (7.39)$$

This follows from [106, Proposition 4.11] with \mathcal{A} and \mathcal{J} there replaced by the current meaning. As a consequence, we get that the factor algebra \mathcal{A}/\mathcal{J} is isomorphic to $L^\infty_{\mathbb{S}}$, and the coset $A + \mathcal{J}$ of $A \in \mathcal{A}$ is represented by the function $b \in L^\infty_{\mathbb{S}}$ from (7.39).

Theorem 7.19 *The operator (7.39) is Fredholm iff it is invertible at infinity and b is invertible in L^∞ .*

Proof. If $A \in \mathcal{A}$ is invertible at infinity and b from its representation (7.39) is invertible in L^∞ , then A is Fredholm by Theorem 5.9 (ii) and Lemma 7.18. Conversely, let $A \in \mathcal{A}$ be Fredholm. By Theorem 5.9 (i) and b), together with Lemma 7.17, we get that A is invertible at infinity. It remains to show that b from (7.39) is invertible in L^∞ . To see this, take $B \in L(L^{p,q})$ and $S, T \in K(L^{p,q})$ such that $AB = I + S$ and $BA = I + T$. Then, for every $k \in \mathbb{N}$, we get that

$$P_k M_b B P_k + P_k K B P_k = P_k A B P_k = P_k + P_k S P_k,$$

and hence

$$(P_k M_b P_k)(P_k B P_k) = P_k + S'$$

with $S' = P_k S P_k - P_k K B P_k \in K(L^{p,q}([-k, k]^N))$ by Lemmas 7.18 and 3.16. From the last equality and its symmetric counter-part, we conclude that $M_{P_k b} = P_k M_b P_k$ is Fredholm on $L^{p,q}([-k, k]^N)$, implying that the function $P_k b$ is invertible in $L^\infty([-k, k]^N)$, by a standard argument (see e.g. [106, Lemma 2.42]). Since this holds for every $k \in \mathbb{N}$, we get that b is invertible in L^∞ . ■

As invertibility of b in L^∞ turned out to be necessary for Fredholmness of (7.39), we will now, without loss of generality, suppose that b is invertible, therefore write

$$A = M_b + K = M_b(I + K') \quad \text{with} \quad K' = M_{b^{-1}}K \in \mathcal{J},$$

and then merely study Fredholmness of $I + K'$. For this setting we can show the analogous result of Theorem 7.1.

Theorem 7.20 *For $A = I + K$ with $K \in \mathcal{J}$, the following statements are equivalent.*

- (a) *All limit operators of A are injective on $L^{\infty,q}$ and $\sigma^{\text{op}}(A)$ has an S -dense subset of injective operators on $L^{1,q}$;*
- (b) *All limit operators of A are invertible on one of the spaces $L^{p,q}$ with $p \in \{0\} \cup [1, \infty]$;*
- (c) *All limit operators A_h of A are invertible on all the spaces $L^{p,q}$ with $p \in \{0\} \cup [1, \infty]$ and the inverses are uniformly bounded (in p and h);*
- (d) *A is Fredholm on one of the spaces $L^{p,q}$ with $p \in \{0\} \cup [1, \infty]$;*
- (e) *A is Fredholm on all the spaces $L^{p,q}$ with $p \in \{0\} \cup [1, \infty]$.*

Thus, for every $p \in \{0\} \cup [1, \infty]$ it holds that

$$\begin{aligned} \text{spec}_{\text{ess}}^p(A) &= \bigcup_{A_h \in \sigma^{\text{op}}(A)} \text{spec}^p(A_h) \\ &= \bigcup_{A_h \in \sigma^{\text{op}}(A)} [\text{spec}_{\text{point}}^{\infty}(A_h) \cup \text{spec}_{\text{point}}^1(A_h)] \end{aligned} \quad (7.40)$$

and

$$\begin{aligned} \text{spec}^p(A) &= \text{spec}_{\text{point}}^{\infty}(A) \cup \text{spec}_{\text{point}}^1(A') \cup \text{spec}_{\text{ess}}^p(A) \\ &= \text{spec}_{\text{point}}^1(A) \cup \text{spec}_{\text{point}}^1(A') \cup \text{spec}_{\text{ess}}^p(A). \end{aligned}$$

Proof. We start by showing that A is subject to the conditions in Corollary 5.23. Clearly, $A = I + K$ is contained in the Wiener algebra and it is rich, by (4.19) and since all generators M_b and C_{κ} of \mathcal{A} are rich. Predual X^{\natural} and preadjoint A^{\natural} exist by Lemma 7.17, and $K \in UM(L^{p,q})$ by Lemma 7.18.

The rest of this proof proceeds exactly as that of Theorem 7.1 with one difference: Unlike Theorem 7.1, which rests on Corollary 5.24, we here have an infinite-dimensional space X and therefore we use Corollary 5.23. ■

Example 7.21 The spectra of limit operators of $A = I + K$ can be written down explicitly when $K \in \mathcal{J}$ is composed of convolution operators C_{κ} with $\kappa \in L^1$ and multiplication operators M_b with a slowly oscillating function $b \in L^{\infty}$. Similarly to Section 5.5.2, by the latter we mean that

$$\text{ess sup}_{t \in [-1,1]^N} |b(x+t) - b(x)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad (7.41)$$

In this case, the multiplication operator M_b is rich and all of its limit operators are multiples of the identity. (In a sense, even the reverse statement is true [106, Proposition 3.52].) As a consequence, every limit operator A_h of $A = I + K$ is of the form $I + C_\kappa$ with some $\kappa \in L^1$, in which case the set $\text{spec}^p(A_h)$ is the range of the function $1 + F\kappa$ with F being the Fourier transform on L^1 . \square

One can now proceed similarly to Section 7.1.1 to get rid of the second injectivity condition in Theorem 7.20 (a) and the second point-spectrum in formula (7.40): Clearly, if $N = 1$, then Corollary 5.25 does exactly this for us. Otherwise, if $N \geq 2$, one way to go is to restrict ourselves to symmetric operators in \mathcal{A} . We will illustrate another way, that is restricting the generating multiplication operators M_b of \mathcal{A} to almost periodic ones, i.e. to work with a type of almost periodic functions $b \in L^\infty$.

Let $b \in L^\infty$ and put $c = (b|_{k+[0,1]^N})_{k \in \mathbb{Z}^N} \in E^\infty(L^\infty([0,1]^N))$. From Example 5.36 we know that the following are equivalent:

- The set $\{V_k b : k \in \mathbb{Z}^N\}$ is relatively compact in L^∞ .
- The set $\{V_k c : k \in \mathbb{Z}^N\}$ is relatively compact in $E^\infty(L^\infty([0,1]^N))$.
- c is almost periodic, i.e. $c \in E_{\text{AP}}^\infty(L^\infty([0,1]^N))$.
- M_c is almost periodic (and therefore rich) on $E^p(X)$.

When this is the case, then, recalling Definition 5.31, we say that $b \in L^\infty$ is \mathbb{Z} -almost periodic and write $b \in L_{\mathbb{Z}\text{AP}}^\infty$. The set of \mathbb{Z} -almost periodic functions on \mathbb{R}^N is not to be confused with the much smaller subset L_{AP}^∞ of almost periodic functions on \mathbb{R}^N . Unlike almost periodic functions, the functions in $L_{\mathbb{Z}\text{AP}}^\infty$ do not need to be continuous (recall our discussion from Section 5.5.1).

So let $\mathcal{A}_{\text{AP}} \subset \mathcal{A}$ denote the \mathcal{W} -closure of the smallest algebra in $L(L^{p,q})$ that contains all M_b with $b \in L_{\mathbb{Z}\text{AP}}^\infty$ and all C_κ with $\kappa \in L^1$. Analogously, \mathcal{J}_{AP} be the smallest \mathcal{W} -closed two-sided ideal in \mathcal{A}_{AP} containing all convolution operators C_κ with $\kappa \in L^1$. It is not hard to see that $\mathcal{J}_{\text{AP}} = \mathcal{J} \cap \mathcal{A}_{\text{AP}}$.

As seen before for operators in \mathcal{A} , the study of Fredholmness in \mathcal{A}_{AP} can be reduced to studying operators of the form $I + K$ with $K \in \mathcal{J}_{\text{AP}}$. In this case, in full analogy to Theorem 7.3, we have the following improved version of formula (7.40):

Theorem 7.22 *If $A = I + K$ with $K \in \mathcal{J}_{\text{AP}}$ then, for all $p \in \{0\} \cup [1, \infty]$ and all $A_h \in \sigma^{\text{op}}(A)$,*

$$\text{spec}_{\text{ess}}^p(A) = \text{spec}^p(A) = \text{spec}^p(A_h) = \bigcup_{B \in \sigma^{\text{op}}(A)} \text{spec}_{\text{point}}^\infty(B).$$

Proof. Firstly, A is almost periodic, by Lemma 5.39 and since the generators of \mathcal{A}_{AP} are almost periodic. Secondly, $\sigma^{\text{op}}(K)$ is uniformly Montel, by Lemmas 7.18 and 4.30. Consequently, we may use Theorems 5.44 and 5.40 (iv), which, together with the equivalence of statements (b) and (d) in Theorem 7.20, prove this formula. ■

7.2 Approximation Methods

In this section we will apply our results from Chapter 6 to slowly oscillating operators [111], to operators acting on the space BC of bounded and continuous functions on \mathbb{R}^N [106, 37] and to boundary integral equations that model a variety of concrete physical problems such as free surface water wave problems and 2D [37, 36] and 3D [82] rough surface scattering problems.

7.2.1 The FSM for Slowly Oscillating Operators

Suppose $E = \ell^p(\mathbb{Z}, \mathbb{C})$ with $1 < p < \infty$; that is $N = 1$, $X = \mathbb{C}$. In accordance with [111], we study the finite section method

$$P_n A P_n u_n = P_n b$$

(i.e., we have $\Omega = [-1, 1]$) for slowly oscillating operators A .

Recall from Section 5.5.2 that we call an operator $A \in BDO(E)$ slowly oscillating if it is subject to the equivalent conditions in Proposition 5.45 and that then all of its limit operators have constant diagonals, i.e. they are Laurent operators. Also recall the notations from Section 5.4: We abbreviate $P_{\mathbb{N}_0} =: P$ and $I - P =: Q$, and we denote by $\text{ind}_+(A)$ the Fredholm index of $A_+ := PAP + Q$ and by $\text{ind}_-(A)$ the Fredholm index of $A_- := QAQ + P$. From Proposition 6.26 and Corollary 6.5 we get the following.

Lemma 7.23 *If $A \in BDO(E)$ is slowly oscillating then the finite section method is applicable to A if and only if A and all operators*

$$QBQ + P \quad \text{and} \quad PCP + Q \quad \text{with} \quad B \in \sigma_+^{\text{op}}(A), \quad C \in \sigma_-^{\text{op}}(A)$$

are invertible.

So, besides the invertibility of A , which is an indispensable ingredient to the applicability of any approximation method to A of course, all we need for the applicability of the finite section method is the invertibility of all operators

$$B_- \quad \text{with} \quad B \in \sigma_+^{\text{op}}(A) \quad \text{and} \quad C_+ \quad \text{with} \quad C \in \sigma_-^{\text{op}}(A), \quad (7.42)$$

where B_- and C_+ are defined by (5.15). From Corollary 5.28 we know that the invertibility of A implies that B_- and C_+ are Fredholm, and

$$\operatorname{ind} B_- = -\operatorname{ind}^+ A \quad \text{and} \quad \operatorname{ind} C_+ = -\operatorname{ind}^- A = \operatorname{ind}^+ A \quad (7.43)$$

for all $B \in \sigma_+^{\text{op}}(A)$ and $C \in \sigma_-^{\text{op}}(A)$. As mentioned before, all limit operators B and C are Laurent operators since A is slowly oscillating (see Proposition 5.46). Consequently, B_- and C_+ are Toeplitz operators on the respective half axes, and for those we have Coburn's theorem (Theorem 2.38 in [21] or Theorem 1.10 in [23]) saying that $\operatorname{ind} B_- = 0$ and $\operatorname{ind} C_+ = 0$ already imply the invertibility of B_- and C_+ .

Summarizing this with (7.43) and the fact that the FSM is applicable if and only if A and all operators (7.42) are invertible, we get:

Proposition 7.24 *If $A \in BDO(E)$ is slowly oscillating then the finite section method is applicable to A if and only if A is invertible and $\operatorname{ind}^+ A = 0$.*

Note that this is a generalisation of the classical result on the stability of the finite section method for band-dominated Laurent operators, which is the case of constant coefficients.

7.2.2 A Special Finite Section Method for BC

In this section we will study an approximation method for operators on the space BC of bounded and continuous functions on \mathbb{R}^N as a Banach subspace of $L^\infty := L^\infty(\mathbb{R}^N)$, where L^∞ is identified via (2.4) with $E^\infty(X)$ for $X = L^\infty([0, 1]^N)$. The operators of interest to us will be of the form

$$A = I + K,$$

where K shall be bounded and linear on L^∞ with the condition $Ku \in BC$ for all $u \in L^\infty$. Typically, K will be some integral operator. One of the simplest examples is a convolution operator $K = C_\kappa$, as introduced in Section 7.1.4, with some $\kappa \in L^1 := L^1(\mathbb{R}^N)$. In this simple case, the validity of the above condition can be easily seen as follows.

Lemma 7.25 *If $\kappa \in L^1$, then $C_\kappa u$ is a continuous function for every $u \in L^\infty$.*

Proof. Let $a = F\kappa$ with some $\kappa \in L^1$. Since κ can be approximated in the norm of L^1 as closely as desired by a continuous function with a compact support, we

may, by Young's inequality [147], suppose that κ already is such a function. But in this case,

$$\begin{aligned} \left| (\mathbf{C}_\kappa u)(x_1) - (\mathbf{C}_\kappa u)(x_2) \right| &= \left| \int_{\mathbb{R}^N} \left(\kappa(x_1 - y) - \kappa(x_2 - y) \right) u(y) \, dy \right| \\ &\leq \|u\|_\infty \int_{\mathbb{R}^N} |\kappa(t + \Delta x) - \kappa(t)| \, dt \end{aligned}$$

clearly tends to zero as $\Delta x = x_1 - x_2 \rightarrow 0$. ■

As a slightly more sophisticated example one could look at an operator of the following form or at the norm limit of a sequence of such operators.

Example 7.26 Put

$$K := \sum_{i=1}^j \mathbf{M}_{b_i} \mathbf{C}_{\kappa_i} \mathbf{M}_{c_i}, \quad (7.44)$$

where $b_i \in BC$, $\kappa_i \in L^1$, $c_i \in L^\infty$ and $j \in \mathbb{N}$. For the condition that K maps L^∞ into BC , it is sufficient to impose continuity of the functions b_i in (7.44), whereas the functions c_i need not be continuous since their action is smoothed by the convolution thereafter. □

We also need the following simple auxiliary result.

Lemma 7.27 *Suppose that $A = I + K$ and that $K \in L(L^\infty)$ and $K(L^\infty) \subset BC$. Abbreviate the restriction $A|_{BC}$ by A_0 . Then the following hold:*

- a) $Au \in BC$ if and only if $u \in BC$;
- b) A is invertible on L^∞ if and only if A_0 is invertible on BC . In this case

$$\|A_0^{-1}\|_{L(BC)} \leq \|A^{-1}\|_{L(L^\infty)} \leq 1 + \|A_0^{-1}\|_{L(BC)} \|K\|_{L(L^\infty)}. \quad (7.45)$$

- c) If A is a Fredholm operator on L^∞ , then A_0 is Fredholm on BC .

Proof. a) This is immediate from $Au = u + Ku$ and $Ku \in BC$ for all $u \in L^\infty$.

b) If A is invertible on L^∞ , then the invertibility of A_0 on BC and the first inequality in (7.45) follows from a).

Now let A_0 be invertible on BC . To see that A is injective on L^∞ , suppose $Au = 0$ for $u \in L^\infty$. From $0 \in BC$ and a) we get that $u \in BC$ and hence $u = 0$ since A is injective on BC . Surjectivity of A on L^∞ : Since A_0 is surjective on BC , for every $v \in L^\infty$ there is a $u \in BC$ such that $A_0 u = Kv \in BC$. Consequently,

$$A(v - u) = Av - A_0 u = v + Kv - Kv = v \quad (7.46)$$

holds, showing the surjectivity of A on L^∞ . So A is invertible on L^∞ , and, by (7.46), $A^{-1}v = v - u = v - A_0^{-1}Kv$ for all $v \in L^\infty$, and hence $A^{-1} = I - A_0^{-1}K$. This proves the second inequality in (7.45).

c) From a) we get that $\ker A \subset BC$ since $0 \in BC$. But this implies that

$$\ker A_0 = \ker A. \quad (7.47)$$

Another immediate consequence of a) is

$$A_0(BC) = A(L^\infty) \cap BC. \quad (7.48)$$

Finally, by (7.48), we have the following relation between factor spaces

$$\frac{BC}{A_0(BC)} = \frac{BC}{A(L^\infty) \cap BC} \cong \frac{BC + A(L^\infty)}{A(L^\infty)} \subset \frac{L^\infty}{A(L^\infty)}. \quad (7.49)$$

So, if A is Fredholm on L^∞ , then (7.47), (7.49) and (7.48) show that also $\ker A_0$ and $BC/A_0(BC)$ are finite-dimensional and $A_0(BC)$ is closed. ■

Remark 7.28 a) The previous lemma clearly holds for arbitrary Banach spaces with one of them contained in the other in place of BC and L^∞ .

b) If, moreover, K has a pre-adjoint operator on L^1 , then an approximation argument as in the proof of Lemma 3.5 of [10] even shows that, in fact, (7.45) can be improved to the equality $\|A_0^{-1}\|_{L(BC)} = \|A^{-1}\|_{L(L^\infty)}$. □

We are concerned with solving the equation $Au = b$ with $A = I + K$, particularly with the case where $u \in L^\infty$ and $b \in BC$ and the equation $Au = b$ is some integral equation

$$u(x) + \int_{\mathbb{R}^N} k(x, y) u(y) dy = b(x), \quad x \in \mathbb{R}^N. \quad (7.50)$$

In this case, by Lemma 7.27 a), we are looking for u in BC only.

In this setting, a popular approximation method which dates back at least to Atkinson [12] and Anselone and Sloan [6], is just to reduce the range of integration from \mathbb{R}^N to the cube $|y| \leq \tau$ for some real number $\tau > 0$. We call this procedure the *finite section method for BC* (short: *BC-FSM*). We are now looking for solutions $u_\tau \in BC$ of

$$u_\tau(x) + \int_{|y| \leq \tau} k(x, y) u_\tau(y) dy = b(x), \quad x \in \mathbb{R}^N \quad (7.51)$$

with $\tau > 0$, and hope that the sequence (u_τ) of solutions of (7.51) strictly converges to the solution u of (7.50) as $\tau \rightarrow \infty$.

This method (7.51) can be written as $A_\tau u_\tau = b$ with

$$A_\tau = I + KP_\tau, \quad (7.52)$$

where $P_\tau \in L(L^\infty)$ is the operator of multiplication by the characteristic function of $[-\tau, \tau]^N$.

As a consequence of Lemma 7.27 a) applied to A_τ , one also has

Corollary 7.29 *For every $\tau > 0$, it holds that $A_\tau u_\tau \in BC$ iff $u_\tau \in BC$.*

In accordance with the machinery presented in Chapter 6, our strategy to study equation (7.50) and the stability of its approximation by (7.51) is to embed these into L^∞ , where we can relate the BC -FSM (7.51) to the usual FSM

$$P_\tau A P_\tau u_\tau = P_\tau b \quad (7.53)$$

on L^∞ .

Remark 7.30 Recall our Remark 6.2: We have restricted our exposition in Chapter 6 to approximating sequences (A_n) with index $n \in \mathbb{N}$ in order to keep things simple. In [103, 104, 106] analogous results for the FSM (7.53) with cut-off parameter $\tau \in (0, \infty)$ have been established, which we find useful for the current section.

As in the setting of a discrete index, we say that a sequence of operators $(A_\tau)_{\tau>0}$ is *stable* if there exists a $\tau_0 > 0$ such that all A_τ with $\tau > \tau_0$ are invertible and their inverses are uniformly bounded. \square

Indeed, the applicabilities of these different methods turn out to be equivalent.

Proposition 7.31 *For the operator $A = I + K$ with $K(L^\infty) \subset BC$, let*

$$A_\tau := I + KP_\tau \quad \text{and} \quad A_{[\tau]} := P_\tau A P_\tau + Q_\tau, \quad \tau > 0.$$

Then the following statements are equivalent.

- (i) *The BC -FSM (A_τ) alias (7.51) is applicable in BC .*
- (ii) *The BC -FSM (A_τ) alias (7.51) is applicable in L^∞ .*
- (iii) *The FSM $(A_{[\tau]})$ is applicable in L^∞ .*
- (iv) *(A_τ) is stable on BC .*
- (v) *(A_τ) is stable on L^∞ .*
- (vi) *$(A_{[\tau]})$ is stable on L^∞ .*

Proof. The implication $(i) \Rightarrow (iv)$ is standard. The equivalence of (iv) and (v) follows from Lemma 7.27 b) applied to A_τ . The equivalence of (v) and (vi) was already pointed out in [112]; it comes from the following observation:

$$\begin{aligned} A_\tau &= I + KP_\tau = P_\tau + P_\tau KP_\tau + Q_\tau + Q_\tau KP_\tau \\ &= P_\tau AP_\tau + Q_\tau(I + Q_\tau KP_\tau) \\ &= (P_\tau AP_\tau + Q_\tau)(I + Q_\tau KP_\tau) \\ &= A_{[\tau]}(I + Q_\tau KP_\tau), \end{aligned}$$

where the second factor $(I + Q_\tau KP_\tau)$ is always invertible with its inverse equal to $I - Q_\tau KP_\tau$, and hence $\|(I + Q_\tau KP_\tau)^{-1}\| \leq 1 + \|K\|$ for all $\tau > 0$.

$(v) \Rightarrow (ii)$. Since (v) implies (vi) , it also implies the invertibility of A on L^∞ by Theorem 4.2 in [106]. But this, together with (v) , implies (ii) by Theorem 1.75 in [106].

Finally, the implication $(ii) \Rightarrow (i)$ is trivial if we keep in mind Lemma 7.27 a) and Corollary 7.29, and the equivalence of (iii) and (vi) follows from Theorem 4.2 in [106]. ■

For the study of property (iii) in Proposition 7.31 we have the continuous version of our Theorem 6.15, which is Theorem 4.2 in [106] (or Theorem 5.2 in [104]), involving limit operators of A , provided that, in addition, A is a rich operator.

In Example 7.26 we would therefore require the functions b_i and c_i to be in $L^\infty_{\mathfrak{S}}$. By [106, Proposition 3.39], this is equivalent to $b_i \in BUC = BC \cap L^\infty_{\mathfrak{S}}$ and $c_i \in L^\infty_{\mathfrak{S}}$ with BUC denoting the space of bounded and uniformly continuous functions on \mathbb{R}^N .

7.2.3 Boundary Integral Equations on Unbounded Rough Surfaces

In accordance with [36, 37], we are now looking at both Fredholmness and BC -FSM for a class of boundary integral equations originating from a variety of physical problems (see e.g. Examples 7.32 – 7.34 below) including the scattering of acoustic waves by an unbounded sound-soft rough surface.

The boundary integral equation method is very well developed as a tool for the analysis and numerical solution of strongly elliptic boundary value problems in both bounded and unbounded domains, provided the boundary itself is bounded (e.g. [13, 117, 161]).

In the case when both domain and boundary are unbounded, the theory of the boundary integral equation method is much less well developed. The reason

for this is fairly clear, namely that loss of compactness of the boundary leads to loss of compactness of boundary integral operators. To be more precise, classical applications of the boundary integral method, for example to potential theory in smooth bounded domains, lead to second kind boundary integral equations of the form $Au = b$ where the function b is known, u unknown, and the operator A is a compact perturbation of the identity (e.g. [13]). In more sophisticated applications, to more complex strongly elliptic systems or to piecewise smooth or general Lipschitz domains, compactness arguments continue to play an important role. For example a standard method to establish that a boundary integral operator A is Fredholm of index zero is to show a Gårding inequality, i.e. to establish that A , as an operator on some Hilbert space, is a compact perturbation of an elliptic principal part (e.g. [117]). The case when the boundary is unbounded is difficult because this tool of compactness is no longer available.

To compensate for loss of compactness, only a few alternative tools are known. In the case of classical potential theory and some other strongly elliptic systems, invertibility and/or Fredholmness of boundary integral operators can be established via direct a priori bounds, using Rellich-type identities. In the context of boundary integral equation formulations these arguments were first systematically exploited by Jerison and Kenig [85], Verchota [177] and Dahlberg and Kenig [50] (and see [88, 118]). The main objective in these papers is to overcome loss of compactness associated with non-smoothness rather than unboundedness of the boundary, but the Rellich identity arguments used are applicable also when the boundary is infinite in extent, notably, and most straightforwardly, when the boundary is the graph of a Lipschitz function. For example, for classical potential theory, invertibility of the operator $A = I + K$, where I is the identity and K the classical double-layer potential operator, can be established when the boundary is the graph of a Lipschitz function, as discussed in [50, 88, 118]. The Rellich-identity estimates establish invertibility of A in the first instance in L^2 , but, by combining these L^2 estimates with additional arguments, the invertibility of A also in L^p for $2 - \varepsilon < p < \infty$ can be established [50, 88]. Here ε is some positive constant which depends only on the space dimension and the Lipschitz constant of the boundary.

The same methods of argument can be extended to some other elliptic problems and elliptic systems, e.g. [61, 119, 120]. Recently L^2 solvability has also been established for a second kind integral equation formulation on the (unbounded) graph of a bounded Lipschitz function in a case (the Dirichlet problem for the Helmholtz equation with real wave number) when the associated weak formulation of the boundary value problem is non-coercive [34, 173]. (This lack of coercivity is relatively easily dealt with as a compact perturbation when the boundary is Lipschitz and compact (e.g. [174]), but is much more problematic

when the boundary is unbounded.)

Here in our approach, we consider the application of the limit operator method as another tool which is available for the study of integral equations on unbounded domains. The results we obtain are applicable to the boundary integral equation formulation of many strongly elliptic boundary value problems in unbounded domains of the form

$$D = \{(x, z) \in \mathbb{R}^N \times \mathbb{R} : z > f(x)\}, \quad (7.54)$$

where $N \geq 1$ and $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is a given bounded and continuous function, $f \in BC$, so that the unbounded boundary is the graph of some bounded function. The results we prove are relevant to the case where the boundary is fairly smooth (Lyapunov), that is f is differentiable with a bounded and α -Hölder continuous gradient for some $\alpha \in (0, 1]$; i.e., for some constant $C > 0$,

$$|\nabla f(x) - \nabla f(y)| \leq C|x - y|^\alpha$$

holds for all $x, y \in \mathbb{R}^N$. This restriction to relatively smooth boundaries has the implication, for many boundary integral operators on ∂D , for example the classical double-layer potential operator (see our Example 7.32 below), that loss of compactness arises from the unboundedness of ∂D rather than its lack of smoothness. To be precise, the boundary integral operators we consider, while not compact are nevertheless locally compact (and hence Montel, see Lemma 7.18), and this local compactness will play a key role in the results we obtain. Throughout we let

$$f_+ = \sup_{x \in \mathbb{R}^N} f(x) \quad \text{and} \quad f_- = \inf_{x \in \mathbb{R}^N} f(x)$$

denote the highest and the lowest elevation of the infinite boundary ∂D . It is convenient to assume, without loss of generality, that $f_- > 0$, so that D is entirely contained in the half space $H = \{(x, z) \in \mathbb{R}^N \times \mathbb{R} : z > 0\}$.

Let us introduce the particular class of second kind integral equations on \mathbb{R}^N that we consider in this section. As we will make clear through detailed examples, equations of this type arise naturally when many strongly elliptic boundary value problems in the domain D are reformulated as boundary integral equations on ∂D . To be specific, boundary value problems arising in acoustic scattering problems [41, 42, 40, 35], in the scattering of elastic waves [7, 8], and in the study of unsteady water waves [132], have all been reformulated as second kind boundary integral equations which, after the obvious parametrization, can be written as

$$u + Ku = v,$$

where K is the integral operator

$$(Ku)(x) = \int_{\mathbb{R}^N} k(x, y) u(y) dy, \quad x \in \mathbb{R}^N \quad (7.55)$$

with kernel k . Further, in all the above examples, the kernel k has the following particular structure which will be the focus of our study, that

$$k(x, y) = \sum_{i=1}^j b_i(x) k_i(x - y, f(x), f(y)) c_i(y), \quad (7.56)$$

where

$$b_i \in BC, \quad k_i \in C((\mathbb{R}^N \setminus \{0\}) \times [f_-, f_+]^2) \quad \text{and} \quad c_i \in L^\infty \quad (7.57)$$

for $i = 1, \dots, j$, and

$$|k(x, y)| \leq \kappa(x - y), \quad x, y \in \mathbb{R}^N, \quad (7.58)$$

for some $\kappa \in L^1$. We note that (7.55)–(7.58) imply that $K \in L(L^p)$ for $1 \leq p \leq \infty$ with $\|K\|_{L(L^p)} \leq \|\kappa\|_1$. In particular,

$$\|K\|_{L(L^\infty)} = \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |k(x, y)| dy \leq \|\kappa\|_1,$$

and we note that $Ku \in BC$ for $u \in L^\infty$.

In the cases cited above, the structure (7.56)–(7.57) is a simple consequence of the invariance with respect to translations in the plane \mathbb{R}^N of the fundamental solutions used in the integral equation formulations. This property follows in turn from invariance in the \mathbb{R}^N plane of the coefficients in the differential operator. In each case the bound (7.58) follows from the Hölder continuity of f , which ensures that $k(x, y)$ is only weakly singular at $x = y$, and from the particular choice of fundamental solution used in the integral equation formulation (a Green's function for the half-plane H in each case), which ensures that $k(x, y)$ decreases sufficiently rapidly as $|x - y| \rightarrow \infty$. Throughout, we will denote the set of all operators K satisfying (7.55)–(7.58) for a particular function $f \in BC$ (but any choices of j and of the functions b_i , k_i , c_i and κ) by \mathcal{K}_f .

There are two main aims of this investigation. The major aim is to apply results from our limit operator method to operators satisfying (7.55)–(7.58), to address, at least partially, two questions for the operator $A = I + K$: Fredholmness and applicability of the BC -FSM. The second aim, of interest in its own right and helpful to the aim of applying our limit operator results, is to relate operators in

the class \mathcal{K}_f , for some $f \in BC$, to classes of integral operators that have been studied previously (see our Section 7.1.4).

Although many of our results can be extended to other function spaces, especially to L^p for $1 \leq p \leq \infty$, we will concentrate on the case where we view A as an operator on BC .

We would like to note that there exist tools which are related to the limit operator method which have been developed by Chandler-Wilde and his collaborators for studying invertibility and the stability and convergence of approximation methods for integral equations on unbounded domains (see [43, 10, 46] and the references therein). These methods can be and have been applied to boundary integral equations of the class that we consider in this section [41, 42, 40, 35, 7, 8, 132]. We note, however, that no systematic study of operators of the class \mathcal{K}_f has been made in these papers. Moreover, the results in these papers are complementary to those we exhibit here: in particular they lead to sufficient but not necessary conditions for invertibility and applicability of the finite section method, and do not provide criteria for Fredholmness.

Examples

We start with some concrete physical problems that have been modelled as elliptic boundary value problem and reformulated as second kind boundary integral equations, the integral operator in each case exhibiting the structure (7.55)-(7.58).

Example 7.32 – Potential Theory. In [132] Preston, Chamberlain and Chandler-Wilde consider the two-dimensional Dirichlet boundary value problem: Given $\varphi_0 \in BC(\partial D)$, find $\varphi \in C^2(D) \cap BC(\overline{D})$ such that

$$\begin{aligned} \Delta \varphi &= 0 & \text{in } D, \\ \varphi &= \varphi_0 & \text{on } \partial D, \end{aligned}$$

which arises in the theory of classical free surface water wave problems. In this case $N = 1$ and the authors suppose that f is differentiable with bounded and α -Hölder continuous first derivative for some $\alpha \in (0, 1]$, i.e., for some constant $C > 0$, $|f'(x) - f'(y)| \leq C|x - y|^\alpha$ for $x, y \in \mathbb{R}$.

Now let

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}) - \Phi(\mathbf{x}^r, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

denote the Green's function for the half plane H where

$$\Phi(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{y}|, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

with $|\cdot|_2$ denoting the Euclidean norm in \mathbb{R}^2 , is the standard fundamental solution for Laplace's equation in two dimensions, and $\mathbf{x}^r = (x_1, -x_2)$ is the reflection of $\mathbf{x} = (x_1, x_2)$ with respect to ∂H . For the solution of the above boundary value problem the following double layer potential ansatz is made in [132]:

$$\varphi(\mathbf{x}) = \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \tilde{u}(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in D,$$

where $\mathbf{n}(\mathbf{y}) = (f'(y), -1)$ is a vector normal to ∂D at $\mathbf{y} = (y, f(y))$, and the density function $\tilde{u} \in BC(\partial D)$ is to be determined. In [132] it is shown that φ satisfies the above Dirichlet boundary value problem if and only if

$$(I - K)\tilde{u} = -2\varphi_0, \quad (7.59)$$

where

$$(K\tilde{u})(\mathbf{x}) = 2 \int_{\partial D} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \tilde{u}(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \partial D. \quad (7.60)$$

In accordance with the parametrization $\mathbf{x} = (x, f(x))$ of ∂D , we define

$$u(x) := \tilde{u}(\mathbf{x}) \quad \text{and} \quad b(x) := -2\varphi_0(\mathbf{x}), \quad x \in \mathbb{R},$$

and rewrite equation (7.59) as the equation

$$u(x) - \int_{-\infty}^{+\infty} k(x, y) u(y) dy = b(x), \quad x \in \mathbb{R}, \quad (7.61)$$

on the real axis for the unknown function $u \in BC(\mathbb{R})$, where

$$\begin{aligned} k(x, y) &= 2 \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} \sqrt{1 + f'(y)^2} = -\frac{1}{\pi} \left(\frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} - \frac{(\mathbf{x}^r - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x}^r - \mathbf{y}|^2} \right) \\ &= -\frac{1}{\pi} \left(\frac{(x - y)f'(y) - f(x) + f(y)}{(x - y)^2 + (f(x) - f(y))^2} - \frac{(x - y)f'(y) + f(x) + f(y)}{(x - y)^2 + (-f(x) - f(y))^2} \right) \\ &= -\frac{1}{\pi} \left(\frac{x - y}{(x - y)^2 + (f(x) - f(y))^2} - \frac{x - y}{(x - y)^2 + (f(x) + f(y))^2} \right) f'(y) \\ &\quad + \frac{1}{\pi} \left(\frac{f(x) - f(y)}{(x - y)^2 + (f(x) - f(y))^2} + \frac{f(x) + f(y)}{(x - y)^2 + (f(x) + f(y))^2} \right). \end{aligned}$$

Clearly $k(x, y)$ is of the form (7.56) with $j = 2$ and property (7.57) satisfied. From Lemma 2.1 and inequality (5) in [132] we moreover get that the inequality (7.58) holds with

$$\kappa(x) = \begin{cases} c|x|^{\alpha-1} & \text{if } 0 < |x| \leq 1, \\ c|x|^{-2} & \text{if } |x| > 1, \end{cases}$$

where $\alpha \in (0, 1]$ is the Hölder exponent of f' , and c is some positive constant. \square

Example 7.33 – Wave scattering by an unbounded rough surface. In [42] Chandler-Wilde, Ross and Zhang consider the corresponding problem for the Helmholtz equation in two dimensions. Given $\varphi_0 \in BC(\partial D)$, they seek $\varphi \in C^2(D) \cap BC(\overline{D})$ such that

$$\begin{aligned}\Delta\varphi + k^2\varphi &= 0 && \text{in } D, \\ \varphi &= \varphi_0 && \text{on } \partial D,\end{aligned}$$

and such that φ satisfies an appropriate radiation condition and constraints on growth at infinity. Again, $N = 1$ and the surface function f is assumed to be differentiable with a bounded and α -Hölder continuous first derivative for some $\alpha \in (0, 1]$. This problem models the scattering of acoustic waves by a sound-soft rough surface; the same problem arises in time-harmonic electromagnetic scattering by a perfectly conducting rough surface.

The authors reformulate this problem as a boundary integral equation which has exactly the form (7.59)-(7.60), except that $G(\mathbf{x}, \mathbf{y})$ is now defined to be the Green's function for the Helmholtz equation in the half-plane H which satisfies the impedance condition $\partial G/\partial x_2 + ikG = 0$ on ∂H . As in Example 7.32, this boundary integral equation can be written in the form (7.61) with $k(x, y)$ of the form (7.56) with $j = 2$ and property (7.57) satisfied, and also here inequality (7.58) holds with

$$\kappa(x) = \begin{cases} c|x|^{\alpha-1} & \text{if } 0 < |x| \leq 1, \\ c|x|^{-3/2} & \text{if } |x| > 1, \end{cases}$$

where $\alpha \in (0, 1]$ is the Hölder exponent of f' , and c is some positive constant.

Note that it is condition (7.58) that fails to hold if one increases the dimension of the scattering problem to three, leading to a boundary integral equation in dimension $N = 2$. For a stable truncation method (as in Section 6.4) that approximately solves the 3D scattering problem by an unbounded rough surface, see our Section 7.2.4 below. \square

Example 7.34 – Wave propagation over a flat inhomogeneous surface. The propagation of mono-frequency acoustic or electromagnetic waves over flat inhomogeneous terrain has been modelled in two dimensions by the Helmholtz equation

$$\Delta\varphi + k^2\varphi = 0$$

in the upper half plane $D = H$ (so $f \equiv 0$ in (7.54)) with a Robin (or impedance) condition

$$\frac{\partial\varphi}{\partial x_2} + ik\beta\varphi = \varphi_0$$

on the boundary line ∂D . Here k , the wavenumber, is constant, $\beta \in L^\infty(\partial D)$ is the surface admittance describing the local properties of the ground surface ∂D , and the inhomogeneous term φ_0 is in $L(\partial D)$ as well.

Similarly to Example 7.33, in fact using the same Green's function $G(\mathbf{x}, \mathbf{y})$ for the Helmholtz equation, Chandler-Wilde, Rahman and Ross [40] reformulate this problem as a boundary integral equation on the real line,

$$u(x) - \int_{-\infty}^{+\infty} \tilde{\kappa}(x-y)z(y)u(y) dy = \psi(x), \quad x \in \mathbb{R}, \quad (7.62)$$

where $\psi \in BC$ is given and $u \in BC$ is to be determined. The function $\tilde{\kappa}$ is in $L^1 \cap C(\mathbb{R} \setminus \{0\})$, and $z \in L^\infty$ is closely connected with the surface admittance β by $z = i(1 - \beta)$.

Note that the kernel function of the integral operator in (7.62) is of the form (7.56) with $j = 1$. The validity of (7.57) and (7.58) is trivial in this case. \square

The Relationship Between \mathcal{K}_f and Other Classes of Integral Operators

Recall that, for a given function $f \in BC$, \mathcal{K}_f denotes the class of all operators K subject to (7.55)–(7.58). For technical reasons we find it convenient to embed the class \mathcal{K}_f into a somewhat larger Banach algebra of integral operators. (It will turn out that this Banach algebra actually is not that much larger than \mathcal{K}_f).

Therefore, given $f \in BC$, put $f_- := \inf f$, $f_+ := \sup f$, and let R_f denote the set of all operators of the form

$$(Bu)(x) = \int_{\mathbb{R}^N} k(x-y, f(x), f(y)) u(y) dy, \quad x \in \mathbb{R}^N \quad (7.63)$$

with $k \in C(\mathbb{R}^N \times [f_-, f_+]^2)$ compactly supported. Moreover, put

$$\begin{aligned} \hat{\mathcal{B}} &:= \text{closspan} \{ M_b B M_c : b \in BC, B \in R_f, c \in L^\infty \}, \\ \mathcal{B} &:= \text{closalg} \{ M_b B M_c : b \in BC, B \in R_f, c \in L^\infty \}, \\ \hat{\mathcal{C}} &:= \text{closspan} \{ M_b C_\kappa M_c : b \in BC, \kappa \in L^1, c \in L^\infty \}, \\ \mathcal{C} &:= \text{closalg} \{ M_b C_\kappa M_c : b \in BC, \kappa \in L^1, c \in L^\infty \}, \\ \mathcal{A} &:= \text{closalg} \{ M_b, C_\kappa : b \in L^\infty, \kappa \in L^1 \}. \end{aligned}$$

Remark 7.35 a) Here, $\text{closspan } M$ denotes the closure in $L(BC)$ of the set of all finite sums of elements of $M \subset L(BC)$, and $\text{closalg } M$ denotes the closure in $L(BC)$ of the set of all finite sum-products of elements of M . So $\text{closspan } M$ is the smallest closed subspace and $\text{closalg } M$ the smallest (not necessarily unital) Banach subalgebra of $L(BC)$ containing M . In both cases we say they are *generated by* M .

b) The following proposition shows that $\hat{\mathcal{B}}$ and \mathcal{B} do not depend on the function $f \in BC$ which is why we omit f in their notations.

c) It is easily seen (see Example 7.26) that all operators in $\hat{\mathcal{C}}$ map arbitrary elements from L^∞ into BC . Consequently, every $K \in \hat{\mathcal{C}}$ is subject to the condition on K in Subsection 7.2.2.

d) Since the generators of \mathcal{A} are band-dominated and since the set of band-dominated operators on $E = L^\infty \cong \ell^\infty(\mathbb{Z}^N, L^\infty([0, 1]^N))$ is a Banach algebra, we get that $\mathcal{A} \subset BDO(E)$.

e) The linear space $\hat{\mathcal{C}}$ is the closure of the set of operators considered in Example 7.26. The following proposition shows that this set already contains all of \mathcal{K}_f . More precisely, it coincides with the closure of \mathcal{K}_f in the norm of $L(BC)$ and with the other spaces and algebras introduced above. \square

Proposition 7.36 *The identity*

$$\text{clos } \mathcal{K}_f = \hat{\mathcal{B}} = \hat{\mathcal{C}} = \mathcal{B} = \mathcal{C} \subset \mathcal{A}$$

holds. In particular, all operators $K \in \mathcal{K}_f$ are band-dominated.

Proof. Clearly, $\hat{\mathcal{C}} \subset \hat{\mathcal{B}}$ since \mathbf{C}_κ with $\kappa \in L^1$ can be approximated in the operator norm by convolutions $B = \mathbf{C}_{\kappa'}$ with a continuous and compactly supported κ' . But these operators B are clearly in R_f .

For the reverse inclusion, $\hat{\mathcal{B}} \subset \hat{\mathcal{C}}$, it is sufficient to show that the generators of $\hat{\mathcal{B}}$ are contained in $\hat{\mathcal{C}}$. We will prove this by showing that $B \in \hat{\mathcal{C}}$ for all $B \in R_f$. So let $k \in C(\mathbb{R}^N \times [f_-, f_+]^2)$ be compactly supported, and define B as in (7.63). To see that $B \in \hat{\mathcal{C}}$, take $L \in \mathbb{N}$, choose $f_- = s_1 < s_2 < \dots < s_{L-1} < s_L = f_+$ equidistant in $[f_-, f_+]$, and let φ_ξ denote the standard hat function for this mesh that is centered at s_ξ (i.e. $\varphi_\xi \in C([f_-, f_+])$ is a linear polynomial on each interval $[s_\eta, s_{\eta+1}]$, $\eta = 1, \dots, L-1$, and $\varphi_\xi(s_\eta) = 1$ if $\xi = \eta$, and $= 0$ if $\xi \neq \eta$). Then, since k is uniformly continuous, its piecewise linear interpolations (with respect to the variables s and t),

$$k^{(L)}(r, s, t) := \sum_{\xi, \eta=1}^L k(r, s_\xi, s_\eta) \varphi_\xi(s) \varphi_\eta(t), \quad r \in \mathbb{R}^N, \quad s, t \in [f_-, f_+],$$

uniformly approximate k as $L \rightarrow \infty$, whence the corresponding integral operators with k replaced by $k^{(L)}$ in (7.63),

$$(B^{(L)}u)(x) = \int_{\mathbb{R}^N} \sum_{\xi, \eta=1}^L k(x - y, s_\xi, s_\eta) \varphi_\xi(f(x)) \varphi_\eta(f(y)) u(y) dy, \quad (7.64)$$

converge to B in the operator norm in $L(BC)$ as $L \rightarrow \infty$. But it is obvious from (7.64) that $B^{(L)} \in \hat{\mathcal{C}}$, which proves that also $B \in \hat{\mathcal{C}}$.

To see that $\mathcal{B} = \mathcal{C}$, it is sufficient to show that the generators of each of the algebras are contained in the other algebra. But this follows from $\hat{\mathcal{B}} = \hat{\mathcal{C}}$, which is already proven.

That \mathcal{C} is contained in the Banach algebra \mathcal{A} generated by L^1 -convolutions and L^∞ -multiplications, is obvious.

For the inclusion $\mathcal{C} \subset \hat{\mathcal{C}}$ it is sufficient to show that $C_\kappa M_b C_\lambda \in \hat{\mathcal{C}}$ for all $\kappa, \lambda \in L^1$ and $b \in L^\infty$. Arguing as in the first paragraph of the proof, it is sufficient to consider the case where κ and λ are continuous and compactly supported, say $\kappa(x) = \lambda(x) = 0$ if $|x| > \ell$. It is now easily checked that

$$(C_\kappa M_b C_\lambda u)(x) = \int_{\mathbb{R}^N} k(x, y) u(y) dy, \quad x \in \mathbb{R}^N,$$

with

$$k(x, y) = \int_{\mathbb{R}^N} \kappa(x - z) b(z) \lambda(z - y) dz = \int_{|t| \leq \ell} \kappa(t) b(x - t) \lambda(x - t - y) dt.$$

By taking a sufficiently fine partition into measurable subsets, $\{T_1, \dots, T_N\}$, of $\{t : |t| < \ell\}$, that is a partition with $\max_i \text{diam } T_i$ sufficiently small, and fixing $t_m \in T_m$ for $m = 1, \dots, N$, we can approximate $k(x, y)$ arbitrarily closely in the supremum norm on $\mathbb{R}^N \times \mathbb{R}^N$ by

$$k(x, y) = \sum_{m=1}^N \int_{T_m} \kappa(t) b(x - t) \lambda(x - t - y) dt \approx k_N(x, y)$$

where

$$\begin{aligned} k_N(x, y) &:= \sum_{m=1}^N \kappa(t_m) \lambda(x - t_m - y) \int_{T_m} b(x - t) dt \\ &= \sum_{m=1}^N \kappa_m \lambda_m(x - y) b_m(x), \quad x, y \in \mathbb{R}^N, \end{aligned} \quad (7.65)$$

with $\kappa_m = \kappa(t_m)$, $\lambda_m(x) = \lambda(x - t_m)$ and $b_m(x) = \int_{T_m} b(x - t) dt$, the latter depending continuously on x . In particular, choosing the partition so that $\max_i \text{diam } T_i < 1/N$, and noting that $k(x, y) = k_N(x, y) = 0$ for $|x - y| > 2\ell$, we see that

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |k(x, y) - k_N(x, y)| dy \rightarrow 0 \text{ as } N \rightarrow \infty,$$

so that $C_\kappa M_b C_\lambda \in \hat{\mathcal{C}}$.

The inclusion $\hat{\mathcal{B}} \subset \text{clos } \mathcal{K}_f$ is also obvious since (7.57) and (7.58) hold if $b_i \in BC$, $c_i \in L^\infty$ and k_i is compactly supported and continuous on all of $\mathbb{R}^N \times [f_-, f_+]^2$.

So it remains to show that $\text{clos } \mathcal{K}_f \subset \hat{\mathcal{B}}$. This clearly follows if we show that $\mathcal{K}_f \subset \hat{\mathcal{B}}$. So let $K \in \mathcal{K}_f$ be arbitrary, that means K is an integral operator of the form (7.55) with a kernel $k(.,.)$ subject to (7.56), (7.57) and (7.58). For every $\ell \in \mathbb{N}$, let $p_\ell : [0, \infty) \rightarrow [0, 1]$ denote a continuous function with support in $[1/(2\ell), 2\ell]$ which is identically equal to 1 on $[1/\ell, \ell]$. Then, for $i = 1, \dots, j$,

$$k_i^{(\ell)}(r, s, t) := p_\ell(|r|) k_i(r, s, t), \quad r \in \mathbb{R}^N, s, t \in [f_-, f_+],$$

is compactly supported and continuous on $\mathbb{R}^N \times [f_-, f_+]^2$, whence $B_i^{(\ell)} \in R_f$ with

$$(B_i^{(\ell)}u)(x) := \int_{\mathbb{R}^N} k_i^{(\ell)}(x - y, f(x), f(y)) u(y) dy, \quad x \in \mathbb{R}^N,$$

for all $u \in BC$. Now put

$$\begin{aligned} k^{(\ell)}(x, y) &:= \sum_{i=1}^j b_i(x) k_i^{(\ell)}(x - y, f(x), f(y)) c_i(y) \\ &= p_\ell(|x - y|) k(x, y), \end{aligned}$$

and let $K^{(\ell)}$ denote the operator (7.55) with k replaced by $k^{(\ell)}$; that is

$$K^{(\ell)} = \sum_{i=1}^j M_{b_i} B_i^{(\ell)} M_{c_i}, \quad (7.66)$$

which is clearly in $\hat{\mathcal{B}}$. It remains to show that $K^{(\ell)} \rightrightarrows K$ as $\ell \rightarrow \infty$. Therefore, note that

$$\begin{aligned} \|K - K^{(\ell)}\| &\leq \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |k(x, y) - k^{(\ell)}(x, y)| dy \\ &= \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |(1 - p_\ell(|x - y|)) k(x, y)| dy \\ &\leq \sup_{x \in \mathbb{R}^N} \int_{|x-y| < 1/\ell} |k(x, y)| dy + \sup_{x \in \mathbb{R}^N} \int_{|x-y| > \ell} |k(x, y)| dy \\ &\leq \sup_{x \in \mathbb{R}^N} \int_{|x-y| < 1/\ell} |\kappa(x - y)| dy + \sup_{x \in \mathbb{R}^N} \int_{|x-y| > \ell} |\kappa(x - y)| dy \\ &= \int_{|z| < 1/\ell} |\kappa(z)| dz + \int_{|z| > \ell} |\kappa(z)| dz \end{aligned}$$

with $\kappa \in L^1$ from (7.58). But clearly, this goes to zero as $\ell \rightarrow \infty$. ■

The Limit Operators of Integral Operators in \mathcal{K}_f

In order to apply our results on Fredholmness and the finite section method to $A = I + K$, we need to know about the limit operators of A , which, clearly, reduces to finding the limit operators of $K \in \mathcal{K}_f$. But before we start looking for these limit operators, we single out a subclass $\mathcal{K}_f^{\mathfrak{s}}$ of \mathcal{K}_f all elements of which are rich operators. So, this time, given $f \in BUC$, let $\mathcal{K}_f^{\mathfrak{s}}$ denote the set of all operators K subject to (7.55)–(7.58), for some $j \in \mathbb{N}$ and $\kappa \in L^1$, with $b_i \in BUC$ and $c_i \in L_{SC\mathfrak{s}}^\infty$ (recall (7.38) and the following sentences) for $i = 1, \dots, j$, and let

$$\begin{aligned}\hat{\mathcal{B}}_{\mathfrak{s}} &:= \text{closspan}\{ M_b B M_c : b \in BUC, B \in R_f, c \in L_{SC\mathfrak{s}}^\infty \}, \\ \mathcal{B}_{\mathfrak{s}} &:= \text{closalg}\{ M_b B M_c : b \in BUC, B \in R_f, c \in L_{SC\mathfrak{s}}^\infty \}, \\ \hat{\mathcal{C}}_{\mathfrak{s}} &:= \text{closspan}\{ M_b C_\kappa M_c : b \in BUC, \kappa \in L^1, c \in L_{SC\mathfrak{s}}^\infty \}, \\ \mathcal{C}_{\mathfrak{s}} &:= \text{closalg}\{ M_b C_\kappa M_c : b \in BUC, \kappa \in L^1, c \in L_{SC\mathfrak{s}}^\infty \}\end{aligned}$$

denote the rich counterparts of $\hat{\mathcal{B}}, \mathcal{B}, \hat{\mathcal{C}}$ and \mathcal{C} . Moreover, put

$$\mathcal{A}'_{\mathfrak{s}} := \text{closalg}\{ M_b, C_\kappa M_c : b \in L_{\mathfrak{s}}^\infty, \kappa \in L^1, c \in L_{SC\mathfrak{s}}^\infty \}.$$

Recall that, by [106, Proposition 3.39], $BC \cap L_{\mathfrak{s}}^\infty = BUC$, and that, by definition of $L_{SC\mathfrak{s}}^\infty$, $C_\kappa M_c$ is rich for all $\kappa \in L^1$ and $c \in L_{SC\mathfrak{s}}^\infty$, whence every operator in $\mathcal{A}'_{\mathfrak{s}}$ is rich. Then the following “rich version” of Proposition 7.36 holds.

Proposition 7.37 *If $f \in BUC$, then it holds that*

$$\text{clos } \mathcal{K}_f^{\mathfrak{s}} = \hat{\mathcal{B}}_{\mathfrak{s}} = \hat{\mathcal{C}}_{\mathfrak{s}} = \mathcal{B}_{\mathfrak{s}} = \mathcal{C}_{\mathfrak{s}} \subset \mathcal{A}'_{\mathfrak{s}}.$$

In particular, every $K \in \mathcal{K}_f^{\mathfrak{s}}$ is rich and band-dominated.

Proof. All we have to check is that the arguments we made in the proof of Proposition 7.36 preserve membership of b and c in BUC and $L_{SC\mathfrak{s}}^\infty$, respectively. In only two of these arguments are there multiplications by b and c involved at all.

The first one is the proof of the inclusion $\hat{\mathcal{B}} \subset \hat{\mathcal{C}}$. In this argument, we show that every $B \in R_f$ is contained in $\hat{\mathcal{C}}$. But in fact, this construction even yields $B \in \hat{\mathcal{C}}_{\mathfrak{s}}$, which can be seen as follows. $B \in R_f$ is approximated in the operator norm by the operators $B^{(L)}$ from (7.64). Since the Courant hats φ_ξ and φ_η are in BUC and also $f \in BUC$, we get $\varphi_\xi \circ f \in BUC$ and $\varphi_\eta \circ f \in BUC \subset L_{SC\mathfrak{s}}^\infty$. So $B^{(L)} \in \hat{\mathcal{C}}_{\mathfrak{s}}$, and hence $B \in \hat{\mathcal{C}}_{\mathfrak{s}}$.

The second argument involving multiplication operators is the proof of the inclusion $\mathcal{C} \subset \hat{\mathcal{C}}$. But also at this point it is easily seen that the functions $b_m(x) = \int_{T_m} b(x-t) dt$ that are invoked in (7.65) are in fact in BUC , whence $\mathcal{C}_{\mathfrak{s}} \subset \hat{\mathcal{C}}_{\mathfrak{s}}$. ■

Now we are ready to say something about the limit operators of $K \in \mathcal{K}_f^\mathfrak{s}$. Not surprisingly, the key to these operators is the behaviour of the surface function f and of the multipliers b_i and c_i at infinity. We will show that every limit operator K_h of K is of the same form (7.55) but with f , b_i and c_i replaced by $f^{(h)}$, $b_i^{(h)}$ and $\tilde{c}_i^{(h)}$, respectively, in (7.56), where we use the notations introduced in and right after (7.37) and (7.38). We will even formulate and prove the analogous result for operators in $\mathcal{B}_\mathfrak{s}$. The key step to this result is the following lemma.

Lemma 7.38 *Let $B \in R_f$; that is, B is of the form (7.63) with a compactly supported kernel function $k \in C(\mathbb{R}^N \times [f_-, f_+]^2)$, and let $c \in L_{\text{SC}\mathfrak{s}}^\infty$. If a sequence $h = (h(m)) \subset \mathbb{Z}^N$ tends to infinity and the functions $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, then the limit operator $(BM_c)_h$ exists and is the integral operator*

$$\left((BM_c)_h u\right)(x) = \int_{\mathbb{R}^N} k(x-y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}^{(h)}(y) u(y) dy, \quad x \in \mathbb{R}^N. \quad (7.67)$$

Proof. Choose $\ell > 0$ large enough that $k(r, s, t) = 0$ for all $r \in \mathbb{R}^N$ with $|r| \geq \ell$ and all $s, t \in [f_-, f_+]$. Now take a sequence $h = (h(m)) \subset \mathbb{Z}^N$ such that the functions $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, i.e. such that

$$\|f|_{h(m)+U} - f^{(h)}|_U\|_\infty \rightarrow 0 \quad \text{and} \quad \|c|_{h(m)+U} - \tilde{c}^{(h)}|_U\|_1 \rightarrow 0 \quad (7.68)$$

as $m \rightarrow \infty$ for every compact set $U \subset \mathbb{R}^N$, which is possible since $f \in BUC \subset L_\mathfrak{s}^\infty$ and $c \in L_{\text{SC}\mathfrak{s}}^\infty$ (see formulas (7.37) and (7.38) and the surrounding text). Then it is easily seen that

$$(V_{-h(m)} BM_c V_{h(m)} u)(x) = \int_{\mathbb{R}^N} k(x-y, f(x+h(m)), f(y+h(m))) c(y+h(m)) u(y) dy$$

for all $x \in \mathbb{R}^N$ and $u \in BC$. Abbreviating $A_m := V_{-h(m)} BM_c V_{h(m)} - (BM_c)_h$, we get that $(A_m u)(x) = \int_{\mathbb{R}^N} d_m(x, y) u(y) dy$, where

$$\begin{aligned} & |d_m(x, y)| \\ &= \left| k(x-y, f(x+h(m)), f(y+h(m))) c(y+h(m)) \right. \\ &\quad \left. - k(x-y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}^{(h)}(y) \right| \\ &\leq \left| k(x-y, f(x+h(m)), f(y+h(m))) - k(x-y, f^{(h)}(x), f^{(h)}(y)) \right| \cdot \|c\|_\infty \\ &\quad + \|k\|_\infty \cdot \left| c(y+h(m)) - \tilde{c}^{(h)}(y) \right| \end{aligned} \quad (7.69)$$

for all $x, y \in \mathbb{R}^N$ and $m \in \mathbb{N}$. Moreover, $d_m(x, y) = 0$ if $|x-y| \geq \ell$.

Now take an arbitrary $\tau > 0$, and denote by U and V the balls around the origin with radius $\tau + \ell$ and τ , respectively. Then, by (7.69),

$$\|P_\tau A_m\| = \operatorname{ess\,sup}_{x \in V} \int_{\mathbb{R}^N} |d_m(x, y)| dy = \operatorname{ess\,sup}_{x \in V} \int_U |d_m(x, y)| dy \rightarrow 0$$

as $m \rightarrow \infty$ since (7.68) holds and k is uniformly continuous. Analogously,

$$\|A_m P_\tau\| = \operatorname{ess\,sup}_{x \in \mathbb{R}^N} \int_V |d_m(x, y)| dy = \operatorname{ess\,sup}_{x \in U} \int_V |d_m(x, y)| dy \rightarrow 0$$

as $m \rightarrow \infty$. This proves that $(BM_c)_h$ from (7.67) is indeed the limit operator of BM_c with respect to the sequence $h = (h(m))$. ■

Proposition 7.39 a) *Let $K = M_b BM_c$ with $b \in BUC$, $B \in R_f$ and $c \in L_{SC\mathfrak{S}}^\infty$. If $h = (h(m)) \subset \mathbb{Z}^N$ tends to infinity and all functions $b^{(h)}$, $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, then the limit operator K_h exists and is the integral operator*

$$(K_h u)(x) = \int_{\mathbb{R}^N} b^{(h)}(x) k(x - y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}^{(h)}(y) u(y) dy, \quad x \in \mathbb{R}^N. \quad (7.70)$$

b) *Every limit operator of $K = M_b BM_c$ with $b \in BUC$, $B \in R_f$ and $c \in L_{SC\mathfrak{S}}^\infty$ is of this form (7.70).*

c) *Formula (7.70) for the limit operators of the generators of the Banach algebra $\mathcal{B}_\mathfrak{S}$ determines all limit operators of every operator $K \in \mathcal{B}_\mathfrak{S}$ in the sense of (4.19). In particular, all limit operators K_h of $K \in \mathcal{K}_f^\mathfrak{S} \subset \mathcal{B}_\mathfrak{S}$, i.e. K given by (7.55)–(7.58), with $j \in \mathbb{N}$, $b_i \in BUC$ and $c_i \in L_{SC\mathfrak{S}}^\infty$ for $i = 1, \dots, j$, are of the same form (7.55) with k replaced by*

$$\hat{k}^{(h)}(x, y) = \sum_{i=1}^j b_i^{(h)}(x) k_i(x - y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}_i^{(h)}(y). \quad (7.71)$$

Proof. **a)** From basic properties of limit operators, (4.19), we get that K_h exists and is equal to $(M_b)_h (BM_c)_h$ which is exactly (7.70) by Lemma 7.38.

b) Suppose $g \subset \mathbb{Z}^N$ is a sequence tending to infinity that leads to a limit operator K_g of K . Since $b, f \in L_\mathfrak{S}^\infty$ and $c \in L_{SC\mathfrak{S}}^\infty$, there is a subsequence h of g such that the functions $b^{(h)}$, $f^{(h)}$ and $\tilde{c}^{(h)}$ exist. But then we are in the situation of a), and the limit operator K_h of K exists and is equal to (7.70). Since h is a subsequence of g , we have $K_g = K_h$.

c) If $K \in \mathcal{B}_\mathfrak{S}$, then K is the norm limit of a sequence of finite sum-products of operators of the form $M_b BM_c$ with $b \in BUC$, $B \in R_f$ and $c \in L_{SC\mathfrak{S}}^\infty$. Enumerate

these operators of the form $M_b B M_c$ (the ones that K decomposes to) by K_ι with $\iota \in J$, where J is an at most countable index set. Now if $g \subset \mathbb{Z}^N$ is any sequence going to infinity such that K_g exists, then, since all operators $K_\iota \in \mathcal{B}_\S$ are rich by Proposition 7.37, we can, by a Cantor diagonal argument, pass to an infinite subsequence h of g such that all the limit operators $(K_\iota)_h$ with $\iota \in J$ exist. Then, by (4.19), the limit operator K_h exists and is composed from the limit operators $(K_\iota)_h$, given by (7.70), in the natural way. But since $h \subset g$, this limit operator K_h equals K_g .

The formula for the limit operators of $K \in \mathcal{K}_f^\S$ follows from the approximation of K by (7.66) for which we explicitly know the limit operators. ■

Example 7.40 Suppose $K \in \mathcal{K}_f^\S$ where the surface function f and the functions b_i and c_i are all slowly oscillating in the sense of (7.41). Let $h \subset \mathbb{Z}^N$ be a sequence tending to infinity such that $b_i^{(h)}$, $f^{(h)}$ and $\tilde{c}_i^{(h)}$ exist – otherwise pass to a subsequence of h with this property which is always possible.

Then, in analogy to Section 5.5.2 (see Section 3.4.5 in [106]), we have that all of $b_i^{(h)}$, $f^{(h)}$ and $\tilde{c}_i^{(h)} = c_i^{(h)}$ are constant. Then, by Proposition 7.39 c), the limit operator K_h is the integral operator with kernel function

$$\hat{k}^{(h)}(x, y) = \sum_{i=1}^j b_i^{(h)} \tilde{c}_i^{(h)} k_i(x - y, f_h^{(h)}, f_h^{(h)}), \quad x, y \in \mathbb{R}^N \quad (7.72)$$

which is just a pure operator of convolution by $\hat{\kappa}^{(h)} \in L^1$ with

$$\hat{\kappa}^{(h)}(x - y) = \hat{k}^{(h)}(x, y) \quad (7.73)$$

for all $x, y \in \mathbb{R}^N$. □

The Main Results

The explicit formula (7.71) for the limit operators of K , given by (7.55)–(7.58), together with our results on Fredholmness and the finite section method in terms of limit operators of A , gives us the desired criteria for Fredholmness and applicability of the BC -FSM of $A = I + K$. These criteria are particularly explicit if all of the functions b_i , c_i and f are slowly oscillating, as in Example 7.40.

- In this case, A is Fredholm on BC if A is Fredholm on L^∞ , which is equivalent to the fact that all Fourier transforms $F\hat{\kappa}_h$ of $\hat{\kappa}_h$ from (7.73) stay away from the point -1 . The latter is necessary for invertibility of A .
- Moreover, it will turn out that the BC -FSM is applicable to A if and only if A is invertible and all functions $F\hat{\kappa}_h$ stay away from -1 and have winding number zero with respect to -1 .

Here are the results in the more general case, for $f \in BUC$ and $K \in \mathcal{K}_f^{\mathfrak{s}}$, so that K is given by (7.55)–(7.58), for some $j \in \mathbb{N}$ with $b_i \in BUC$ and $c_i \in L_{\text{SC}\mathfrak{s}}^\infty$ for $i = 1, \dots, j$.

From Proposition 7.37 we know that $K \in \mathcal{A}'_{\mathfrak{s}}$. By (4.19), all limit operators of $A = I + K$ are of the form $A_h = I + K_h$, i.e., by Proposition 7.39 c),

$$(A_h u)(x) = u(x) + \int_{\mathbb{R}^N} \sum_{i=1}^j b_i^{(h)}(x) k_i(x - y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}_i^{(h)}(y) u(y) dy \quad (7.74)$$

for $u \in BC$ and $x \in \mathbb{R}^N$.

Theorem 7.41 *If $f \in BUC$ and $K \in \mathcal{K}_f^{\mathfrak{s}}$, then the following implications hold for $A = I + K$,*

$$(i) \Leftarrow (ii) \Leftrightarrow (iii) \Leftarrow (iv) \Leftrightarrow (v),$$

where

- (i) A is a Fredholm operator on BC .
- (ii) A is a Fredholm operator on L^∞ .
- (iii) All limit operators (7.74) of A are invertible on L^∞ .
- (iv) A is invertible as an operator on BC .
- (v) A is invertible as an operator on L^∞ .

Proof. By Proposition 7.37, A is rich and band-dominated. As in Lemma 7.17 and 7.18, we see that K has a preadjoint and is Montel. So we get from Theorem 5.9 that $(ii) \Leftrightarrow (iii)$. By Lemma 7.27 b) and c), we get $(i) \Leftarrow (ii)$ and $(iv) \Leftrightarrow (v)$. Finally, $(ii) \Leftarrow (v)$ is trivial. ■

Since $K \in \mathcal{K}_f$ maps L^∞ into BC (see Remark 7.35 c)), we can, by Proposition 7.31, study the applicability of the BC -FSM (7.51) for $A = I + K$ by passing to its FSM (7.53) on L^∞ instead. This method is studied, for the case $N = 1$, in Theorem 4.2 in [106]. So, for simplicity, let us restrict ourselves to operators on the axis, $N = 1$. Higher dimensional results in the style of our Theorem 6.15 but with index $\tau \in (0, \infty)$ can be derived analogously.

By Theorem 4.2 in [106], we have to look at all operators of the form

$$QV_{-\tau}A_hV_\tau Q + P \quad \text{with} \quad A_h \in \sigma_+^{\text{op}}(A) \quad (7.75)$$

and

$$PV_{-\tau}A_hV_\tau P + Q \quad \text{with} \quad A_h \in \sigma_-^{\text{op}}(A) \quad (7.76)$$

with $\tau \in \mathbb{R}$, where $P = M_{\chi_{[0,+\infty)}}$, $Q = I - P$, and $\sigma_{\pm}^{\text{op}}(A)$ refers to the set of limit operators A_h of A with $h(m) \rightarrow \pm\infty$, respectively. The operator (7.75) is invertible if and only if the operator $QV_{-\tau}A_hV_{\tau}Q$, mapping u to

$$u(x) + \int_{-\infty}^0 \sum_{i=1}^j b_i^{(h)}(x+\tau) k_i(x-y, f^{(h)}(x+\tau), f^{(h)}(y+\tau)) \tilde{c}_i^{(h)}(y+\tau) u(y) dy \quad (7.77)$$

for $x < 0$, is invertible on the negative half axis, or, equivalently, if the operator

$$V_{\tau}(QV_{-\tau}A_hV_{\tau}Q)V_{-\tau} = (V_{\tau}QV_{-\tau})A_h(V_{\tau}QV_{-\tau}) = P_{(-\infty, \tau]}A_hP_{(-\infty, \tau]},$$

mapping u to

$$u(x) + \int_{-\infty}^{\tau} \sum_{i=1}^j b_i^{(h)}(x) k_i(x-y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}_i^{(h)}(y) u(y) dy, \quad x < \tau,$$

is invertible on the half axis $(-\infty, \tau)$, for the corresponding sequence h leading to a limit operator at plus infinity.

Completely analogously, the operator (7.76) is invertible if and only if the operator $PV_{-\tau}A_hV_{\tau}P$, mapping u to

$$u(x) + \int_0^{+\infty} \sum_{i=1}^j b_i^{(h)}(x+\tau) k_i(x-y, f^{(h)}(x+\tau), f^{(h)}(y+\tau)) \tilde{c}_i^{(h)}(y+\tau) u(y) dy \quad (7.78)$$

with $x > 0$ is invertible on the positive half axis, or, equivalently, if the operator that maps u to

$$u(x) + \int_{\tau}^{+\infty} \sum_{i=1}^j b_i^{(h)}(x) k_i(x-y, f^{(h)}(x), f^{(h)}(y)) \tilde{c}_i^{(h)}(y) u(y) dy, \quad x > \tau$$

is invertible on the half axis $(\tau, +\infty)$, for the corresponding sequence h leading to a limit operator at minus infinity.

Combining this with our previous results, we get the following theorem. For brevity we will say that a set $\{A_{\tau}\}_{\tau \in \mathbb{R}}$ of operators is *uniformly invertible* if all A_{τ} are invertible and their inverses are uniformly bounded, and we call it *essentially invertible* if almost all (i.e. with exceptions in an index set of measure zero) A_{τ} are invertible and their inverses are uniformly bounded.

Theorem 7.42 *If $f \in BUC$ and $K \in \mathcal{K}_f^{\$}$, then the BC-FSM is applicable to $A = I + K$ if and only if*

- (i) A is invertible on L^∞ ,
- (ii) for every sequence h leading to $+\infty$ for which the limit operator A_h exists, the set of operators $\{QV_{-\tau}A_hV_\tau Q\}_{\tau \in \mathbb{R}} = \{u \mapsto (7.77)\}_{\tau \in \mathbb{R}}$ is essentially invertible on $L^\infty(-\infty, 0)$, and
- (iii) for every sequence h leading to $-\infty$ for which the limit operator A_h exists, the set of operators $\{PV_{-\tau}A_hV_\tau P\}_{\tau \in \mathbb{R}} = \{u \mapsto (7.78)\}_{\tau \in \mathbb{R}}$ is essentially invertible on $L^\infty(0, +\infty)$.

Proof. Combine Proposition 7.31 above and Theorem 4.2 in [106]. ■

Remark 7.43 a) Both the operators $QV_{-\tau}A_hV_\tau Q$ and $PV_{-\tau}A_hV_\tau P$ depend continuously (with respect to the operator norm on $L^\infty(-\infty, 0)$ and $L^\infty(0, +\infty)$, respectively) on $\tau \in \mathbb{R}$. This implies that each ‘essentially invertible’ can be replaced by ‘uniformly invertible’ in the above theorem.

b) If, as in Example 7.40, all of f , b_i and c_i are slowly oscillating, then we have $A_h = I + C_{F\hat{\kappa}^{(h)}}$ with $\hat{\kappa}^{(h)}$ as introduced in Example 7.40. In this case, by Theorem 7.41, A is Fredholm if -1 is not in the spectrum of any $C_{F\hat{\kappa}^{(h)}}$; that is, all the (closed, connected) curves $F\hat{\kappa}^{(h)}(\dot{\mathbb{R}}) \subset \mathbb{C}$ stay away from the point -1 . (Here $\dot{\mathbb{R}}$ stands for the one-point compactification $\mathbb{R} \cup \{\infty\}$ of the real line. Note that $F\hat{\kappa}^{(h)}(\infty) = 0$, by the Riemann-Lebesgue lemma.) Moreover, the BC -FSM is applicable to A if and only if A is invertible and all curves $F\hat{\kappa}^{(h)}(\dot{\mathbb{R}})$, in addition to staying away from -1 , have winding number zero with respect to this point.

c) In some cases (see Example 7.44 below) the functions $\hat{k}^{(h)}(x, y)$ from (7.72) in Example 7.40 even depend on $|x - y|$ only, which shows that the same is true for $\hat{\kappa}^{(h)}(x - y) := \hat{k}^{(h)}(x, y)$ then. If we then look at the applicability of the BC -FSM for $N = 1$, we get the following interesting result: The invertibility of A already implies the applicability of the finite section method (also cf. [111]). Indeed, if A is invertible then all limit operators A_h are invertible, which shows that all functions $F\hat{\kappa}_h$ stay away from the point -1 . But from $F\hat{\kappa}^{(h)}(z) = F\hat{\kappa}^{(h)}(-z)$ for all $z \in \mathbb{R}$ we get that the point $F\hat{\kappa}^{(h)}(z)$ traces the same curve (just in opposite directions) for $z < 0$ and for $z > 0$. So the winding number of the curve $F\hat{\kappa}^{(h)}(\dot{\mathbb{R}})$ around -1 is automatically zero. □

Example 7.44 Let us come back to Example 7.32 where, as we found out earlier, $N = 1$, $j = 2$, $b_1 \equiv -1/\pi$, $c_1 = f'$, $b_2 \equiv 1/\pi$, $c_2 \equiv 1$,

$$k_1(r, s, t) = \frac{r}{r^2 + (s - t)^2} - \frac{r}{r^2 + (s + t)^2}$$

and

$$k_2(r, s, t) = \frac{s - t}{r^2 + (s - t)^2} + \frac{s + t}{r^2 + (s + t)^2}.$$

In addition, suppose that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$. Then, by Lemma 3.45 b) in [106], all of b_1, b_2, c_1, c_2 and f are slowly oscillating, and, for every sequence h leading to infinity such that the strict limit $f^{(h)}$ exists, we have that $b_1^{(h)} \equiv -1/\pi$, $c_1^{(h)} \equiv 0$, $b_2^{(h)} \equiv 1/\pi$, $c_2^{(h)} \equiv 1$, and $f^{(h)} \geq f_- > 0$ is a constant function, whence

$$\begin{aligned} \hat{k}^{(h)}(x, y) &= \frac{1}{\pi} \left(\frac{f^{(h)} - f^{(h)}}{(x-y)^2 + (f^{(h)} - f^{(h)})^2} + \frac{f^{(h)} + f^{(h)}}{(x-y)^2 + (f^{(h)} + f^{(h)})^2} \right) \\ &= \frac{2f^{(h)}}{\pi} \frac{1}{(x-y)^2 + 4(f^{(h)})^2} =: \hat{\kappa}^{(h)}(x-y), \quad x, y \in \mathbb{R}^N \end{aligned}$$

where $f^{(h)}$ is an accumulation value of f at infinity.

Now it remains to check the function values of the Fourier transform $F\hat{\kappa}^{(h)}$. A little exercise in contour integration shows that $F\hat{\kappa}^{(h)}(z) = \exp(-2f^{(h)}|z|)$ for $z \in \mathbb{R}$ (cf. Remark 7.43 d)). So $F\hat{\kappa}^{(h)}(\mathbb{R})$ stays away from -1 and has winding number zero.

Consequently, by our criteria derived earlier, we get that A is Fredholm and that the finite section method is applicable if and only if A is invertible.

As discussed in [132], by other, somewhat related arguments, it can, in fact, be shown that A is invertible, even when f is not slowly oscillating. Precisely, injectivity of A can be established via applications of the maximum principle to the associated BVP, and then limit operator-type arguments can be used to establish surjectivity.

We note also that the modified version of the finite section method proposed in [116] could be applied in this case. (This method first approximates the actual surface function f by a function f_τ for which f'_τ is compactly supported and $f_\tau(s) = f(s)$ for $|s| \leq \tau - \tau^*$, and then applies the finite section method (7.51). Here $\tau^* \in (0, \tau)$ is some parameter whose value is fixed independently of τ , for all τ sufficiently large.) For this modified version the arguments of [116] and the invertibility of A establish applicability even when f is not slowly oscillating, provided τ^* is chosen large enough. \square

7.2.4 Rough Surface Scattering in 3D

In accordance with [82], we now study the approximate solution of a boundary integral equation from 3D rough surface scattering using the rectangular truncation method introduced in Section 6.4.

As in the previous section, the problem is the solution of integral equations over unbounded domains which appear in what is usually called the rough surface scattering problem in the engineering literature. It denotes scattering of acoustic

or electromagnetic waves by a surface which is a non-local perturbation of an infinite plane surface. We will restrict our attention to the case where the scattering surface

$$\Gamma = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\}$$

is the graph of some bounded continuous function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. The domain of wave propagation is the upper perturbed half-space

$$D := \{x = (x_1, x_2, x_3) : x_3 > f(x_1, x_2)\}.$$

We will assume that $f \in C^{1,\alpha}$, $\alpha \in (0, 1)$, is a Lyapunov function, i.e. it is continuously differentiable with Hölder continuous first derivative.

Rough surface scattering problems arise in important applications for acoustic and electromagnetic waves. Well-known areas are outdoor sound propagation or optical scattering in nano-technology. We refer to the extensive literature reviewed in [126], [182], [158], [183] and [57].

Recently, Chandler-Wilde, Heinemeyer and Potthast [33], [34] provided some rigorous existence theory for the integral equation approach in three dimensions. But to our knowledge there is no rigorous numerical analysis for the solution of these integral equations in three dimensions. This is the starting point of our work. However, the approach presented here is based only on very general properties of the operator equations under consideration. We expect that it can be used for a whole range of different problems, and it will be the starting point for further research.

If we regard our integral equation as an equation

$$A\varphi = b \tag{7.79}$$

on the space $E = L^2(\mathbb{R}^2)$, for example, then recall that the finite section method (FSM) consists in replacing (7.79) by

$$P_\varrho A P_\varrho \varphi = P_\varrho b, \tag{7.80}$$

where $\varrho > 0$. For simplicity, we will assume in this section that the projection operator $P_\varrho : E \rightarrow E$ is given by

$$(P_\varrho \psi)(x) := \begin{cases} \psi(x), & |x| < \varrho \\ 0, & \text{otherwise,} \end{cases} \tag{7.81}$$

so that P_ϱ truncates a function on \mathbb{R}^2 to its values inside the bounded set

$$B_\varrho := \{x \in \mathbb{R}^2 : |x| < \varrho\},$$

where $|\cdot|$ denotes an arbitrary norm in \mathbb{R}^2 .

So the FSM (7.80) restricts right-hand side b , solution φ and resulting function $A\varphi$ all to the same set B_ϱ . Of course, the idea is, assuming equation (7.79) is uniquely solvable for every right-hand side b , to hope that also equation (7.80) is uniquely solvable (now in $L^2(B_\varrho)$, of course) and that its solution φ approximates the exact solution φ_0 of (7.79) if only one chooses ϱ large enough.

In Chapter 6 we have given sufficient and necessary criteria on the applicability of the FSM for rich band-dominated operators A , and in Section 7.2.3 we have applied these results to integral equations (7.55) subject to (7.56) – (7.58) which includes Examples 7.32 – 7.34. In the 3D scattering problem at hand, the integral kernel does not have property (7.58) which makes it more difficult to apply our theorems on the FSM (or already to check their conditions). So we came up with the following idea:

In contrast to exactly solving the truncated equation (7.80) for large ϱ , we propose to look for a function $\varphi \in E$ with

$$P_\varrho A P_\tau \varphi \approx P_\varrho b \quad (7.82)$$

for large ϱ, τ and a given discrepancy allowance δ in the ' \approx ' sign. So we have two main differences to the finite section method:

- (a) We allow two different cut-off parameters ϱ and τ instead of just one.
- (b) We work with approximate instead of exact solutions.

From the matrix perspective, point (a) means that we cut rectangular rather than quadratic finite matrices out of the original infinite matrix that represents our operator A – whence we will call the method *rectangular finite section method* or rFSM.

We will show that for every precision $\delta > 0$ there are positive constants ϱ, τ such that there exists a function $\varphi \in Y$ which satisfies the conditions

$$P_\tau \varphi = \varphi \quad (7.83)$$

$$\|P_\varrho A P_\tau \varphi - P_\varrho b\|_E \leq \delta, \quad (7.84)$$

i.e. φ is supported on B_τ and it satisfies the truncated approximate equation (7.82) on B_ϱ .

In our main convergence result we show that, given any $\varepsilon > 0$, we can choose the parameters δ, ϱ, τ such that every solution φ of the rFSM (7.83)–(7.84) approximates the true solution φ_0 of the original equation (7.79) with

$$\|\varphi - \varphi_0\|_E \leq \varepsilon.$$

Remark 7.45 We would like to clarify the relation between the rFSM and the theory of regularization of ill-posed problems. Clearly, in general, the truncated equation $P_\delta A P_\tau \varphi = P_\delta b$ does not have a solution and, thus, describes an ill-posed equation. In the theory of inverse problems such equations are solved approximately by using a family R_α of bounded linear inversion operators which (for injective operators) tend pointwise to the inverse of the equation under consideration, see [60]. However, here we are not interested in the case where $\delta \rightarrow 0$ while ρ and τ are fixed, which would be the case interesting in the framework of inverse problems. Here we keep δ fixed and then choose τ, ρ sufficiently large, i.e. we change the equation which we solve. \square

Scattering by Rough Surfaces

We restrict our attention to time-harmonic acoustic waves, which are modelled by the Helmholtz equation

$$\Delta u + \kappa^2 u = 0. \quad (7.85)$$

Here, κ denotes the wave number, which for the real-valued case is linked to the speed of sound c and the frequency ω via $\kappa = \omega/c > 0$. Often, κ is admitted to be a complex number $\kappa = \kappa_0 + i\sigma$, where the imaginary part σ models the properties of some lossy medium.

For our scattering surface Γ we assume that $f \in BC^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0, 1]$. Further, f is assumed to satisfy the bounds

$$0 < f^- \leq f(x) \leq f^+, \quad x \in \mathbb{R}^2.$$

We consider the scattering of an *incident acoustic wave* u^i by the surface Γ . The *total field* $u := u^i + u^s$ is the sum of the incident field and the *scattered field* u^s . The scattered field is a solution to the Helmholtz equation (7.85) in D . Further, we assume that u satisfies the Dirichlet boundary condition

$$u(x) = 0, \quad x \in \Gamma. \quad (7.86)$$

The scattered field is required to be bounded in D , i.e.

$$|u^s(x)| \leq c, \quad x \in D, \quad (7.87)$$

for some constant c . Further, we follow [34] and require that u satisfies the *limiting absorption principle* as follows. If u is considered in its dependence on the complex wave number κ , we write $u = u^{(\kappa)}$. We assume that, for all sufficiently small $\sigma > 0$, a solution of (7.85), (7.86), (7.87) exists and satisfies

$$u^{(\kappa_0 + i\sigma)} \rightarrow u^{(\kappa_0)}, \quad \sigma \rightarrow 0 \quad (7.88)$$

for all $x \in D$. The limiting absorption principle is a kind of radiation condition which is needed to obtain the physically correct solution to the scattering problem.

The free-space fundamental solution in three dimensions is given by the function

$$\Phi(x, y) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|}, \quad x \neq y \in \mathbb{R}^3.$$

The decay of this function for $|y| \rightarrow \infty$ is very weak and it has not been possible to use it for the solution of scattering problems over unbounded domains in three dimensions. However, in [33] Chandler-Wilde, Heinemeyer and Potthast employ the Green's function for the half-space $\{x \in \mathbb{R}^3 : x_3 > 0\}$

$$G(x, y) := \Phi(x, y) - \Phi(x, y')$$

with $y' := (y_1, y_2, -y_3)$ to derive an integral equation of the second kind for the solution of the rough surface scattering problem. The decay of the function G is given in [33], equation (3.8), as

$$|G(x, y)| \leq \frac{C}{|x - y|^2}, \quad (7.89)$$

for all $x, y \in \mathbb{R}^3$ with $f^- < x_3, y_3 < f^+$ with some constant C . For the normal derivative of G the estimate

$$\left| \frac{\partial G(x, y)}{\partial \nu(y)} \right| \leq \frac{C}{|x - y|^2}, \quad (7.90)$$

holds for all $x, y \in \mathbb{R}^3$ with $f^- < x_3, y_3 < f^+$ with some constant C , compare equation (3.11) in [33].

We are now prepared to formulate the scattering problem under consideration: For the incident field $u^i(x) := \Phi(x, z)$ with $z \in D$ we seek a scattered field $u^s \in C^2(D) \cap C(\overline{D})$ which satisfies the Helmholtz equation (7.85) in D , the Dirichlet boundary condition (7.86) on Γ , the bound (7.87) and the limiting absorption principle (7.88).

The rough surface scattering problem is transformed into a boundary value problem via the Ansatz

$$u^s(x) = v(x) - \Phi(x, z'),$$

where v satisfies the Helmholtz equation (7.85), the bound (7.87), the limiting absorption principle (7.88) and the boundary condition

$$v(x) = g(x), \quad x \in \Gamma \quad (7.91)$$

with

$$g(x) := -(\Phi(x, z) - \Phi(x, z')) = -G(x, z).$$

A solution to the boundary value problem can be found via the single- and double-layer potential approach. We define the *single-layer potential*

$$u_1(x) = \int_{\Gamma} G(x, y) \varphi(y) ds(y), \quad x \in \mathbb{R}^3,$$

and the *double-layer potential*

$$u_2(x) = \int_{\Gamma} \frac{\partial G(x, y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \mathbb{R}^3.$$

The boundary values of these potentials can be calculated using the boundary integral operators S and K defined by

$$(S\varphi)(x) = 2 \int_{\Gamma} G(x, y) \varphi(y) ds(y), \quad x \in \Gamma,$$

and the *double-layer potential*

$$(K\varphi)(x) = 2 \int_{\Gamma} \frac{\partial G(x, y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \Gamma.$$

It is shown in [33] that the combined single- and double layer potential

$$v(x) := u_2(x) - i\eta u_1(x), \quad x \in D, \quad (7.92)$$

with parameter $\eta > 0$ satisfies the boundary value problem (7.91) if and only if the density $\varphi \in L^2(\Gamma)$ satisfies the integral equation

$$(I + K - i\eta S)\varphi = 2g, \quad (7.93)$$

which is of the form $(I - W)\varphi = b$ with $W = -K + i\eta S$ and $b = 2g$. The basic uniqueness and existence result is given by Theorem 3.4 of [34] as follows.

Proposition 7.46 *The operator $I + K - i\eta S$ is boundedly invertible on $L^2(\Gamma)$, and for the norm of its inverse, one has*

$$\|(I + K - i\eta S)^{-1}\| \leq B, \quad (7.94)$$

where the constant B , as given in (3.4) in [34], only depends on the quotient κ/η and the Lipschitz constant of f .

As a result of Proposition 7.46 we obtain the existence of the solution for the rough surface scattering problem. In principle, the solution to the scattering problem is given by the combined potential (7.92) with a density φ which satisfies (7.93). Our main topic here is the numerical solution of such integral equations in three dimensions.

The Rectangular Finite Section Method

We will formulate our rectangular finite section method (from this point abbreviated by rFSM) for the following abstract setting which includes the rough surface scattering problems discussed above.

Let E be a Banach space, and let $\{P_\varrho\}_{\varrho>0}$ be a family of linear operators on E with the following three properties,

- (P1) $P_\varrho P_\tau = P_\tau = P_\tau P_\varrho$ for all $\varrho \geq \tau > 0$,
- (P2) $\|P_\varrho\| = 1$ for all $\varrho > 0$,
- (P3) $P_\varrho \rightarrow I$ strongly, i.e. $P_\varrho \varphi \rightarrow \varphi$ for all $\varphi \in E$, as $\varrho \rightarrow \infty$.

From (P1) with $\varrho = \tau$ we get that every P_ϱ is a projector. We will also have to deal with the complementary projectors $I - P_\varrho$ which, for brevity, shall again be denoted by Q_ϱ , for every $\varrho > 0$.

Now suppose A is a bounded linear operator on E with

- (A1) A is invertible on E ,
- (A2) $\|Q_\varrho A P_\tau\| \rightarrow 0$ as $\varrho \rightarrow \infty$ for every fixed $\tau > 0$.

For illustration we give an example of E , $\{P_\varrho\}_{\varrho>0}$ and A that includes the setting from equation (7.93).

Example 7.47 Let $E = L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$ and $N \in \mathbb{N}$, and define P_ϱ as in (7.81), for every $\varrho > 0$. Then E and the family $\{P_\varrho\}_{\varrho>0}$ are clearly subject to our assumptions (P1)–(P3).

Now let $A = I - W$, where W is a well-defined and bounded integral operator

$$(W\varphi)(x) = \int_{\mathbb{R}^n} k(x, y) \varphi(y) dy, \quad x \in \mathbb{R}^n \quad (7.95)$$

on E . Hence the kernel function $k(\cdot, \cdot)$ must satisfy a decay condition

$$|k(x, y)| \leq \frac{C}{|x - y|^\gamma} \quad \text{for } |x - y| > 1 \quad (7.96)$$

with constants $\gamma > 0$ and some $C > 0$. \square

Note that this example, with $p = N = \gamma = 2$, covers the operators in the boundary integral formulation (7.93) arising from 3D rough surface scattering problems as discussed before. The following lemma shows that our assumption (A2) automatically holds for the operator A from Example 7.47 if $\gamma p > n$.

Lemma 7.48 *Let $A = I - W$ act boundedly on $E = L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$, $N \in \mathbb{N}$ and W as in (7.95) and (7.96) with $C > 0$ and $\gamma > 0$. Then, for every $\tau > 0$, we have*

$$\|Q_\varrho AP_\tau\| \leq \frac{c}{\varrho^{\gamma-n/p}}, \quad \varrho > 2\tau \quad (7.97)$$

with some constant $c > 0$ depending on τ . In particular, if $\gamma p > N$, then assumption (A2) holds.

Proof. Let $B_\varrho = \{x \in \mathbb{R}^N : |x| \leq \varrho\}$ and $\partial B_\varrho = \{x \in \mathbb{R}^N : |x| = \varrho\}$ be the ball and sphere of radius $\varrho > 0$ in \mathbb{R}^N , and denote their N - and $(N-1)$ -dimensional measure by $|B_\varrho|$ and $|\partial B_\varrho|$, respectively. Now take $\tau > 0$ and some $\varrho > 2\tau$ and first suppose $1 < p < \infty$. Using Hölder's inequality with $1/p + 1/q = 1$ we get the following.

$$\begin{aligned} \|Q_\varrho AP_\tau \varphi\|_{L^p(\mathbb{R}^n)}^p &= \int_{|x| \geq \varrho} |(AP_\tau \varphi)(x)|^p dx \\ &= \int_{|x| \geq \varrho} |(WP_\tau \varphi)(x)|^p dx \\ &= \int_{|x| \geq \varrho} \left| \int_{|y| < \tau} k(x, y) \varphi(y) dy \right|^p dx \\ &\leq \int_{|x| \geq \varrho} \left(\left(\int_{|y| < \tau} |k(x, y)|^q dy \right)^{1/q} \cdot \|P_\tau \varphi\|_{L^p(\mathbb{R}^n)} \right)^p dx \\ &\leq \int_{|x| \geq \varrho} \left(\int_{|y| < \tau} |k(x, y)|^q dy \right)^{p/q} dx \cdot \|\varphi\|_{L^p(\mathbb{R}^N)}^p \end{aligned}$$

Consequently, using the bound (7.96) and the inequality

$$|x| \geq \varrho > 2\tau > 2|y|, \quad \text{which implies} \quad |x - y| \geq |x| - |y| > |x|/2,$$

we get

$$\begin{aligned}
\|Q_\varrho AP_\tau\|^p &\leq \int_{|x|\geq\varrho} \left(\int_{|y|<\tau} \frac{C^q}{|x-y|^{\gamma q}} dy \right)^{p/q} dx \\
&\leq \int_{|x|\geq\varrho} \left(\int_{|y|<\tau} \frac{C^q}{(|x|/2)^{\gamma q}} dy \right)^{p/q} dx \\
&= \left(\int_{|x|\geq\varrho} \frac{1}{|x|^{\gamma p}} dx \right) \left(\int_{|y|<\tau} dy \right)^{p/q} 2^{\gamma p} C^p \quad (7.98) \\
&= \left(\int_{r=\varrho}^{+\infty} \frac{1}{r^{\gamma p}} |\partial B_r| dr \right) |B_\tau|^{p/q} 2^{\gamma p} C^p \\
&= \left(\int_{r=\varrho}^{+\infty} \frac{r^{n-1}}{r^{\gamma p}} dr \right) |\partial B_1| |B_\tau|^{p/q} 2^{\gamma p} C^p \\
&= \varrho^{n-\gamma p} \frac{1}{\gamma p - n} |\partial B_1| |B_\tau|^{p/q} 2^{\gamma p} C^p.
\end{aligned}$$

Finally, taking p -th roots proves (7.97). The proof for $p = 1$ is similar. But instead of using Hölder's inequality one immediately arrives at (7.98), with p/q replaced by 0. ■

Further examples of settings in which (P1)–(P3) hold are Example 2.5 and of course our sequence spaces $E = E^p(X)$ with $1 \leq p < \infty$, an arbitrary Banach space X and the associated projectors from Definition 2.2, where we put $P_\varrho := P_{\{-[\varrho], \dots, [\varrho]\}^N}$ for every $\varrho > 0$.

Given a Banach space E and a family of projections $\{P_\varrho\}_{\varrho>0}$ on E with properties (P1)–(P3), an operator A on E with properties (A1) and (A2), and an arbitrary element $b \in E$, we are looking for the (unique) solution $\varphi =: \varphi_0$ of (7.79); that is of

$$A\varphi = b.$$

For the approximate solution of this equation we propose the following method:

For given precision $\delta > 0$ and sufficiently large cut-off parameters ϱ and τ , calculate a solution $\varphi \in E$ of the system

$$\left. \begin{aligned} P_\tau \varphi &= \varphi \\ \|P_\varrho AP_\tau \varphi - P_\varrho b\| &\leq \delta. \end{aligned} \right\} \quad (\text{rFSM})$$

Definition 7.49 *We say that $\tau_0 > 0$ is an admissible τ -bound for A , b and a given precision $\delta > 0$ if (rFSM) is solvable in E for all $\varrho > 0$ and $\tau > \tau_0$.*

Theorem 7.50 *For every $\delta > 0$, there is an admissible τ -bound $\tau_0 = \tau_0(\delta) > 0$.*

Proof. Let $\varrho > 0$ be arbitrary. We demonstrate how to choose τ_0 so that

$$\varphi_1 := P_\tau \varphi_0 = P_\tau A^{-1}b \quad (7.99)$$

solves the system (rFSM) for every $\tau > \tau_0$, with φ_0 being the exact solution of (7.79). Since P_τ is a projector, we have

$$P_\tau \varphi_1 = P_\tau^2 \varphi_0 = P_\tau \varphi_0 = \varphi_1$$

for every $\tau > 0$. Furthermore, for all $\varrho > 0$ and $\tau > 0$, we have

$$\begin{aligned} \|P_\varrho A P_\tau \varphi_1 - P_\varrho b\| &= \|P_\varrho A P_\tau^2 A^{-1}b - P_\varrho b\| \\ &\leq \|P_\varrho A A^{-1}b - P_\varrho b\| + \|P_\varrho A Q_\tau A^{-1}b\| \\ &\leq 0 + \|A\| \cdot \|Q_\tau A^{-1}b\|. \end{aligned}$$

But, by assumption (P3), there is a $\tau_0 > 0$ such that

$$\|Q_\tau A^{-1}b\| \leq \frac{\delta}{\|A\|} \quad (7.100)$$

for all $\tau > \tau_0$, so that finally

$$\|P_\varrho A P_\tau \varphi_1 - P_\varrho b\| \leq \delta$$

holds, and hence φ_1 is a solution of the system (rFSM) for all $\tau > \tau_0$ and $\varrho > 0$. ■

Lemma 7.51 *Let $\tau_0 > 0$ be an admissible τ -bound for a given precision $\delta > 0$. If $\tau > \tau_0$ and $\varrho > 0$ are such that $\|Q_\varrho A P_\tau\| < 1/\|A^{-1}\|$ then the set of all solutions of (rFSM) is a bounded subset of E . Precisely, every solution $\varphi \in E$ of the system (rFSM) is subject to $\|\varphi\|_E \leq M$ with M given by (7.101).*

Proof. Suppose $\varphi \in E$ solves (rFSM) for given parameters $\delta, \varrho, \tau > 0$. Then

$$\begin{aligned} \|A\varphi\| - \|P_\varrho b\| &\leq \|A\varphi - P_\varrho b\| = \|A P_\tau \varphi - P_\varrho b\| \\ &\leq \|A P_\tau \varphi - P_\varrho A P_\tau \varphi\| + \|P_\varrho A P_\tau \varphi - P_\varrho b\| \\ &\leq \|Q_\varrho A P_\tau\| \cdot \|\varphi\| + \delta \end{aligned}$$

together with $\|\varphi\| \leq \|A^{-1}\| \cdot \|A\varphi\|$ implies that

$$\begin{aligned} \frac{\|\varphi\|}{\|A^{-1}\|} &\leq \|A\varphi\| \leq \|P_\varrho b\| + \|Q_\varrho A P_\tau\| \cdot \|\varphi\| + \delta \\ &\leq \|b\| + \|Q_\varrho A P_\tau\| \cdot \|\varphi\| + \delta \end{aligned}$$

and hence

$$\|\varphi\| \leq M := \frac{\|b\| + \delta}{1/\|A^{-1}\| - \|Q_\varrho A P_\tau\|}. \quad (7.101)$$

■

Theorem 7.52 *For every $\varepsilon > 0$, there are parameters δ, ϱ, τ such that every solution $\varphi \in E$ of the system (rFSM) is an approximation*

$$\|\varphi - \varphi_0\|_E < \varepsilon \quad (7.102)$$

of the exact solution φ_0 of (7.79). Precisely, there are functions $\delta_0, \tau_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\varrho_0 : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$ such that if $\delta < \delta_0(\varepsilon)$, $\tau > \tau_0(\delta)$ and $\varrho > \varrho_0(\varepsilon, \delta, \tau)$, then every solution $\varphi \in E$ of (rFSM) is subject to (7.102).

Proof. Let $\varepsilon > 0$ be given. We start the proof with three preliminary steps.

- (a) Choose $\delta < \delta_0 := \frac{\varepsilon}{3\|A^{-1}\|}$.
- (b) Choose $\tau_0 > 0$ such that $(\|Q_\tau \varphi_0\| =)\|Q_\tau A^{-1}b\| \leq \frac{\delta}{\|A\|}$ for all $\tau > \tau_0$, so that τ_0 is an admissible τ -bound for δ (see inequality (7.100)). Now let $\tau > \tau_0$.
- (c) Choose $\varrho_0 > 0$ such that $\|Q_\varrho b\| < \frac{\varepsilon}{3\|A^{-1}\|}$ and

$$\|Q_\varrho AP_\tau\| < \frac{1}{\|A^{-1}\|} \left(1 - \frac{1}{1 + \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|}} \right) \quad (7.103)$$

for all $\varrho > \varrho_0$, and fix some $\varrho > \varrho_0$.

Now let $\varphi \in E$ be a solution of (rFSM) with parameters δ, τ and ϱ as chosen above. From (7.103) we get $\|Q_\varrho AP_\tau\| < 1/\|A^{-1}\|$, and hence, by Lemma 7.51,

$$\|\varphi\| \leq M \quad (7.104)$$

with M as defined in (7.101). Moreover, inequality (7.103) is equivalent to

$$\|Q_\varrho AP_\tau\| < \frac{1}{\|A^{-1}\|} \cdot \frac{\frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|}}{1 + \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|}},$$

and hence to

$$\left(1 + \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|} \right) \cdot \|Q_\varrho AP_\tau\| < \frac{1}{\|A^{-1}\|} \cdot \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|}.$$

This, moreover, is equivalent to

$$\begin{aligned} \|Q_\varrho AP_\tau\| &< \frac{1}{\|A^{-1}\|} \cdot \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|} - \frac{\varepsilon}{3(\|b\| + \delta) \cdot \|A^{-1}\|} \cdot \|Q_\varrho AP_\tau\| \\ &= \frac{\varepsilon(1/\|A^{-1}\| - \|Q_\varrho AP_\tau\|)}{3(\|b\| + \delta) \cdot \|A^{-1}\|} = \frac{\varepsilon}{3M\|A^{-1}\|} \end{aligned} \quad (7.105)$$

with M as defined in (7.101). Then we have

$$\begin{aligned}
 \|\varphi - \varphi_0\| &= \|P_\tau \varphi - \varphi_0\| = \|A^{-1}AP_\tau \varphi - A^{-1}b\| \leq \|A^{-1}\| \cdot \|AP_\tau \varphi - b\| \\
 &\leq \|A^{-1}\| \cdot (\|AP_\tau \varphi - P_\varrho AP_\tau \varphi\| + \|P_\varrho AP_\tau \varphi - P_\varrho b\| + \|P_\varrho b - b\|) \\
 &\leq \|A^{-1}\| \cdot (\|Q_\varrho AP_\tau\| \cdot \|\varphi\| + \delta + \|Q_\varrho b\|) \\
 &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
 \end{aligned}$$

using inequalities (7.105) and (7.104) and the bounds on δ and $\|Q_\varrho b\|$ in the last step. ■

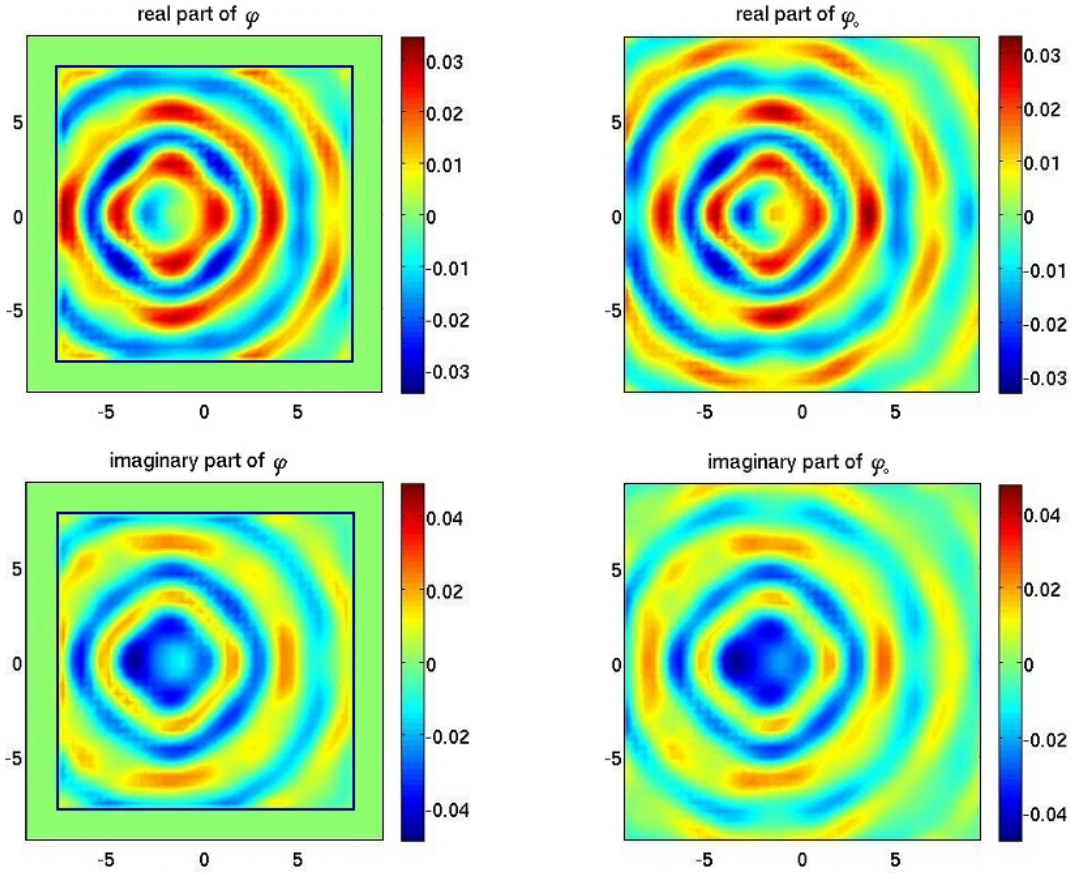


Figure 7.12: The images show, for a particular 3D scattering problem, the real and imaginary part of the density φ solving system (rFSM) in the left column and the true density φ_0 in the right column. On the set B_τ the approximation is visible and on $B_\varrho \setminus B_\tau$ the density φ is zero by construction.

Remark 7.53 One way to effectively solve the system (rFSM) for given parameters ϱ, τ and δ is to compute a $\varphi \in E$ that minimises the discrepancy in (7.82), for example using a gradient method or, if possible, by directly applying the Moore Penrose pseudo-inverse B^+ of $B := P_\varrho A P_\tau$ to the right-hand side $P_\varrho b$.

If E is a Hilbert space then it is well-known that $\varphi \in E$ minimises the residual $\|B\varphi - P_\varrho b\|$ if and only if $B^*(B\varphi - P_\varrho b) = 0$. If, in addition, P_ϱ is self-adjoint for all $\varrho > 0$, then, after re-substituting B , the latter is equivalent to

$$P_\tau A^* P_\varrho A P_\tau \varphi = P_\tau A^* P_\varrho b. \quad (7.106)$$

However, if ϱ is sufficiently large, then, by (A2) and (P3), the equation (7.106) is just a small perturbation of

$$P_\tau A^* A P_\tau \varphi = P_\tau A^* b, \quad (7.107)$$

which is nothing but the finite section method for the equation

$$A^* A \varphi = A^* b. \quad (7.108)$$

Note that the finite section method (7.107) is applicable since $A^* A$ is positive definite (see, e.g. Theorem 1.10 b in [81]). Clearly, if A is invertible, as we require in (A1), then also its adjoint A^* is invertible, and (7.108) is equivalent to our original equation (7.79).

Summarizing, if E is a Hilbert space and all P_ϱ are self-adjoint, then minimising $\|P_\varrho A P_\tau \varphi - P_\varrho b\|$ is equivalent to solving a slight perturbation (7.106) of the finite section method (7.107) for (7.108). \square

The rFSM can be applied to our rough surface scattering problem as we know from Example 7.47, Lemma 7.48 and inequalities (7.89) and (7.90). In particular, note that the bound (7.94) on $\|A^{-1}\|$ enables us to actually compute the corresponding terms in step (a) and (c) in the proof of Theorem 7.52. We summarise our results in the following theorem.

Theorem 7.54 *The rFSM, applied to rough surface scattering (7.85), (7.86), (7.87) and (7.88), is convergent in the sense of Theorem 7.52.*

In [82] one can find numerous calculations to illustrate Theorem 7.54. Here we will restrict ourselves to Figure 7.12, where we visually compare a solution φ of (rFSM) and the exact solution φ_0 of (7.79) for a particular 3D rough surface scattering problem.

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Theses

on the Habilitation thesis

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This text is concerned with the Fredholm theory and stable approximation of bounded linear operators generated by a class of infinite matrices (a_{ij}) that are either banded or have certain decay properties as one goes away from the main diagonal. The operators are studied on ℓ^p spaces, henceforth denoted by $E^p(X)$, of functions $\mathbb{Z}^N \rightarrow X$, where $p \in [1, \infty]$, $N \in \mathbb{N}$ and X is a complex Banach space. The latter means that our matrix entries a_{ij} are indexed by multiindices $i, j \in \mathbb{Z}^N$ and that every a_{ij} is itself a bounded linear operator on X . Our main focus lies on the case $p = \infty$, where new results are derived, and it is demonstrated in both general theory and concrete operator equations from mathematical physics how advantage can be taken of these new $p = \infty$ results in the general case $p \in [1, \infty]$. A central notion in the study of such an operator A is the set $\sigma^{\text{op}}(A)$ of all its so-called limit operators. We now briefly summarise the main results.

Thesis 1. Let $BDO(E)$ denote the set of all band-dominated operators (that is the completion in $L(E)$ of the set of all operators induced by a band matrix) on E and write $UM(E)$ for the set of all $K \in BDO(E)$ that are induced by a matrix (κ_{ij}) for which the set of all its entries is collectively compact on X . Moreover, an operator is called rich if $\sigma^{\text{op}}(A)$ has a certain compactness property.

a) If $A = I + K$ is a rich band-dominated operator on $E = E^\infty(X)$, where $K \in UM(E)$, X has a predual and A has a preadjoint operator then the following are shown to be equivalent:

- (i) A is a Fredholm operator;
- (ii) A is invertible at infinity;
- (iii) all limit operators $B \in \sigma^{\text{op}}(A)$ of A are injective and there is an s -dense subset of $\sigma^{\text{op}}(A)$ of surjective operators.

b) The surjectivity condition in (iii) is shown to be redundant in each of the following cases:

- when $N = 1$,
- when A is an almost periodic operator,
- when A is a slowly oscillating operator,
- when A is a pseudoergodic operator.

Thesis 2. We say that A belongs to the Wiener algebra \mathcal{W} if it is induced by a matrix (a_{ij}) with

$$\|A\|_{\mathcal{W}} := \sum_{k \in \mathbb{Z}^N} \sup_{j \in \mathbb{Z}^N} \|a_{j+k,j}\|_{L(X)} < \infty.$$

In this case, $A \in BDO(E^p(X))$ for all $p \in [1, \infty]$. It has been known [95] that invertibility of such an operator is independent of the value of p in the space $E = E^p(X)$ that it is considered as acting on. We show that the same is true for the Fredholm property and the Fredholm index; precisely:

If $A \in \mathcal{W}$ is Fredholm on one space $E = E^p(X)$ with $p \in [1, \infty)$ and if one of the following conditions holds

- $A = I + K$ is rich and $K \in UM(E)$, or
- X has an isomorphic subspace of codimension one (this is the so-called hyperplane property)

then A is Fredholm on all spaces $E^q(X)$ with $q \in [1, \infty]$ and its Fredholm index is independent of q . The statement extends to $p = \infty$ if X has a predual and A , as an operator on $E^\infty(X)$, has a preadjoint operator.

Thesis 3. Let A be induced by an infinite matrix (a_{ij}) . The finite section method (FSM) consist in replacing the equation $Au = b$ (where $A \in L(E)$ and $b \in E$ are given and $u \in E$ is to be determined), i.e. the infinite linear system

$$\sum_{j \in \mathbb{Z}^N} a_{ij} u(j) = b(i), \quad i \in \mathbb{Z}^N$$

by a finite system $A_n u_n = b_n$ of the form

$$\sum_{j \in \Omega_n} a_{ij} u_n(j) = b(i), \quad i \in \Omega_n, \quad (8.1)$$

where $\Omega_n = (n\Omega) \cap \mathbb{Z}^N$ with a compact set $\Omega \subset \mathbb{R}^N$ and $n \in \mathbb{N}$. The FSM is called applicable if (8.1) is uniquely solvable for all sufficiently large $n \in \mathbb{N}$ and every RHS b and if its solution u_n converges strictly to the solution u of $Au = b$ as $n \rightarrow \infty$.

We show that if $A \in BDO(E)$ is rich and $\Omega \subset \mathbb{R}^N$ is a convex polytope with vertices in \mathbb{Z}^N and 0 in its interior then the FSM is applicable iff A and every operator from an associated set is invertible and the inverses are uniformly bounded.

We give a description of this associated set, we generalise the statement to subsequences (this means that (8.1) is only expected to be uniquely solvable, with solutions convergent to u , for a particular sequence $n = n_1, n_2, \dots$ of natural numbers), and we pass to the more general setting of a starlike set Ω .

Thesis 4. As an alternative approach to the FSM (8.1), we discuss the slightly modified truncation scheme $A_{m,n} u_{m,n} \approx b_m$ of the form

$$\sum_{j \in \Omega_n} a_{ij} u_{m,n}(j) \approx b(i), \quad i \in \Omega_m, \quad (8.2)$$

leading to rectangular instead of quadratic finite subsystems of $Au = b$ that are now to be solved approximately instead of exactly.

We prove that if A is induced by a matrix (a_{ij}) with $a_{ij} \rightarrow 0$ as $|i| \rightarrow \infty$ for every j and if A is invertible then the modified method (8.2) is applicable. By the latter we mean that, for every $\varepsilon > 0$ and every $b \in E$, there exist $m_0, n_0 \in \mathbb{N}$ and a precision $\delta > 0$ such that all (approximate) solutions of the rectangular system $\|A_{m,n} u_{m,n} - b_m\| < \delta$ with $m > m_0$ and $n > n_0$ are in the ε -neighbourhood of the exact solution u of $Au = b$.

We remark that the conditions for the applicability of this method are much weaker and easier to verify than those for the FSM and we discuss how the two truncation parameters m and n are to be coupled.

Selbständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich diese Habilitationsschrift selbständig und unter Verwendung keiner anderen als der angegebenen Hilfsmittel und Quellen verfasst habe.

Chemnitz, den 23. Februar 2009